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A CONSTRUCTION OF THE SUPERCUSPIDAL REPRESENTATIONS OF $\mathrm{GL}_n(F)$, F p-ADIC

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ABSTRACT. Let F be a nondiscrete, locally compact, non-Archimedean field. In this paper, we construct all irreducible supercuspidal representations of $G = \mathrm{GL}_n(F)$. For each such representation π (which we may as well assume is unitary), we give a subgroup J of G that is compact mod the center Z of G and a (finite-dimensional) representation σ of J such that inducing σ to G gives π. The proof that all supercuspidals have been constructed appeals to a theorem (the Matching Theorem) that has been proved by global methods.

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Let F be a p-adic (= locally compact, nondiscrete, non-Archimedean) field. In this paper we prove:

(1.1) **Theorem.** *The irreducible supercuspidal representations of* $\mathrm{GL}_n(F)$ *are all induced from representations of open compact* (mod *center*) *subgroups.*

In fact, we construct a set of inducing representations for the supercuspidals of $\mathrm{GL}_n(F)$. This should make it possible to do further calculations concerning these representations. For example, in this paper we compute their formal degrees explicitly.

The first major breakthrough in constructing supercuspidals for $G = \mathrm{GL}_n(F)$ was made by Howe [10], who gave a construction in the case $p \nmid n$ (the "tamely ramified" case). Moy [19] proved that Howe had indeed constructed all the supercuspidals for these n. Meanwhile, Carayol [3] gave a construction for prime n (including $n = p$). In all these cases the general outline was the same: one first uses the similarities between G and $D_n^\times$ (where D_n is a central division algebra over F with $[D_n : F] = n^2$) to construct a set of supercuspidals for $\mathrm{GL}_n(F)$, and then uses the Matching Theorem of Deligne-Kazhdan-Vigneras [7] (see also [21]) to show that the set is complete. The proof given here uses the same procedure. In using the Matching Theorem, it is necessary to know the number of irreducible representations of $D_n^\times$ with conductor less than a

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fixed number m. This information is provided in [14]. It is therefore not essential in the second step to know $(D_n^\times)^\wedge$, although it is useful. However, the construction of $D_n^\times$ given in [4] is similar to (but simpler than) the one used here. (We remark further in §11 on the logical connections between this paper and [4].)

The procedure described above for showing that one has found all supercuspidals can succeed only for reductive groups of type A_n, because only for such groups is there a compact form of the group. For this reason it is important to have "intrinsic" or "local" proofs of the completeness of the construction. For $\mathrm{GL}_n(F)$, n the product of two primes, Kutzko and Manderscheid [16] have shown that all supercuspidals are induced. More recently, a proof of completeness for the case $(n, p) = 1$ has been given by Howe and Moy [11], and for the construction in [10] (n prime) by them [11] and Bushnell [2]. These rely on the theory of minimal K-types, developed originally by Howe and Moy. The construction in this paper seems well adapted to the minimal K-type picture, and it would not be surprising to see a local proof of completeness in the near future.

The problem of constructing supercuspidals exists, of course, for general reductive p-adic groups. For $\mathrm{GL}_m(D)$, D a local division algebra, the methods of this paper seem to apply with only minor modifications. (These groups are of course of type A_n.) Gerardin [8] gives a construction of some unramified supercuspidals; more recently, Morris [18] has given a construction in a situation like the "very cuspidal" case of Carayol; Moy [20] and Asmuth and Keys [1] have analyzed the situation for $\mathrm{GSp}_4(F)$, Moy in the case $p \neq 2$ and Asmuth and Keys in general. A complete analysis of the general reductive case will probably depend on advances in the theory of minimal K-types.

A very rough idea of the construction is as follows: supercuspidal representations are connected with anisotropic tori, and we should therefore look at maximal compact (mod center) subgroups that contain such tori. In the tamely ramified case (where n is prime to p), one then considers certain characters of the torus that are appropriately nondegenerate. One then determines the elements x of the subgroup such that conjugation by x fixes χ, extends χ to this subgroup, and induces to get the supercuspidal. (In some cases there is an additional step, that tensoring by a finite-dimensional representation like a representation of the Heisenberg group. One may also need to tensor with lifts of cuspidal representations of $\mathrm{GL}_m(k_f)$, where k_f is the residue class field of an unramified extension of F and $m|n$.) The general situation is similar, but the characters are no longer characters on the torus; instead, they are defined on certain subgroups of the compact open subgroup in a way that associates them with the torus less directly. The key property is that conjugation by elements of the torus fixes the character. That is, we concentrate more on the group of elements fixing χ under conjugation than on χ itself. Eventually, we define χ on a compact-mod-center subgroup H such that only elements of H fix χ; we then induce to create the supercuspidal. (Again, there may be the additional step of tensoring with a Heisenberg-like representation or with the lift of a cuspidal representation.)

We now fix some notation in order to give a more detailed description of the construction. Let $n = ef$ and let $V = F^n$; denote by $\mathcal{O}$ the ring of integers of F and by P its (unique) maximal ideal. Define a lattice chain

$\mathscr{L} = \{\dots, L_{-1}, L_0, L_1 \dots\}$ in F^n by

$$\begin{aligned}
&L_0 = \mathscr{O} \oplus \cdots \oplus \mathscr{O} \quad (n \text{ terms}) \\
&L_1 = \mathscr{O} \oplus \cdots \oplus \mathscr{O} \oplus P \oplus \cdots \oplus P \quad (f\ P\text{'s}) \\
&\cdots \\
&L_{e-1} = \mathscr{O} \oplus \cdots \oplus \mathscr{O} \oplus P \oplus \cdots \oplus P \quad (f(e-1)\ P\text{'s}) \\
&L_e = P \oplus \cdots \oplus P, \\
&L_{me+j} = P^m L_j \qquad (0 \le j \le e-1).
\end{aligned}$$

Set

$$A_e^j = \{x \in M_n(F) = xL_i \subseteq L_{i+j}\ \forall i \in \mathbb{Z}\}, \quad j \in \mathbb{Z}; \qquad A_e = A_e^0,$$

$$K_e = \text{group of invertible elements in } A_e,$$

$$K_e^j = 1 + A_e^j, \qquad j \ge 1,$$

$$\varpi_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ \varpi_F & 0 & \cdots & \cdots & 0 \end{bmatrix} \in \mathrm{GL}_n(F) \quad (\varpi_F \text{ generates } P), \tag{1.2}$$

$$\varpi_e = \varpi_n^f, \qquad Z_e = \text{ group generated by } \varpi_e.$$

Then $\varpi_e \colon L_i \to L_{i+1}$ $\forall i$ and Z_eK_e is the normalizer of K_e in $\mathrm{GL}_n(F)$; K_e and the K_e^j are all compact open normal subgroups of Z_eK_e. Furthermore, $\varpi_e^e = \varpi_F I$ is central in $\mathrm{GL}_n(F)$ and the groups Z_eK_e are maximal compact (mod center) subgroups of $\mathrm{GL}_n(F)$. If E is a field extension of F with ramification index e and residue class degree f, then $E^\times$ embeds into Z_eK_e. Therefore each Z_eK_e contains anisotropic maximal tori, and indeed any maximal anisotropic torus is contained in a conjugate of some Z_eK_e. (This description of these groups, using lattices, was introduced into the subject by Bushnell.)

Let $k \subseteq \mathscr{O}$ denote the set of solutions to $X^q - X = 0$, where q is the cardinality of the residue class field $\mathscr{O}/P$. The elements in k give representatives for $\mathscr{O}/P$, and we usually identify the two. (This should not cause confusion.) Then

$$K_e^m/K_e^{m+1} \cong A_e^m/A_e^{m+1} \cong (M_f(k))^e \quad \text{if } m \ge 1, \text{ under } 1 + y \mapsto y;$$
$$K_e/K_e^1 \cong \mathrm{GL}_f(k)^e.$$

Write a typical element of $M_f(k)^e$ as $\alpha = (\alpha_0, \dots, \alpha_{e-1})$, where each $\alpha_i \in M_f(k)$. Then ϖ_e normalizes K_e^m and K_e^{m+1}, and hence induces an action on $M_f(k)^e$. This action is independent of m; in fact,

$$\varpi_e \alpha \varpi_e^{-1} = \alpha^{\sigma_e}, \qquad \alpha^{\sigma_e} = (\alpha_1, \dots, \alpha_{e-1}, \alpha_0).$$

A similar formula holds for K_e/K_e^1. We also note a few facts about the relation of K_e^j and $K_{e'}^{j'}$ when $e'|e$. Let $e = e'e_0$. Then $K_{e'}^{j'} \subseteq K_e^j$ if $(j'-1)e_0 \ge j-1$ and $K_{e'}^{j'} \supseteq K_e^j$ if $j'e_0 \le j$. If $x \in A_{e'}^{j'} \cap A_e^j$, then $\varpi_e x \in A_{e'}^{j'} \cap A_e^{j+1}$. When $j'e_0 = j$, there are coset representatives for K_e^j/K_e^{j+1} that are in $K_{e'}^{j'}$ (mod $K_{e'}^{j'+1}$).

For most of the paper, e (and hence f) will be fixed, and we will suppress it in some notation. Let $m_e = M_f(k)^e$ and $m_e^\times = \mathrm{GL}_f(k)^e$, so that any $y \in Z_e K_e$ can be written as

$$y = \alpha_0 \varpi_e^{j_0}(1 + \alpha_1\varpi_e + \alpha_2\varpi_e^2 + \cdots),$$

where the α_j are preimages in $M_n(\mathscr{O}_F)$ of elements of m_e and α_0 is the preimage of an element of $m_e^\times$. (We say more about the choice of preimages below.) We usually write ϖ for ϖ_e and σ for σ_e.

Let ψ be a fixed additive character of F, trivial on P_F but not on $\mathscr{O}_F$, and define χ_x to be the character of $M_n(F)$ given by

$$\chi_x(y) = \psi \circ \mathrm{Tr}(xy).$$

Then it is standard that $x \mapsto \chi_x$ gives an isomorphism of $M = M_n(F)$ with $M^\frown$ under which $(A_e^s)^\perp = A_e^{1-s}$.

We are going to start with sequences of triples $(s_1, e_1, f_1), \dots, (s_r, e_r, f_r)$ with the following properties:

(1.3)

(i) $s_1 > s_2 > \cdots > s_r \geq 0$;

(ii) $e_1|e_2|\cdots|e_r$, $f_1|f_2|\cdots|f_r$, and $1 < e_1f_1 < \cdots < e_rf_r$;

(iii) $e_r = e$ and $f_r = f$;

(iv) for all i, $f|s_i$ and $e_i(n/e_{i-1}, s_i) = n$ (where $(\ ,\ )$ is the greatest common divisor and $e_0 = 1$);

(v) $e_r = e_{r-1}$ if $s_r = 0$.

We then construct supercuspidal representations π of G associated with these triples.

The construction of π will occupy §§6–9 of this paper. The following is a brief description (more remarks appear early in §6): We want to induce π from a character χ on a subgroup H that we construct. (As noted earlier, this statement sometimes needs to be modified slightly.) Define $H^j = H \cap K_e^j$; write $s_i = ft_i$, and assume that $t_1 > 0$. We will have χ trivial on $K_e^{t_1+1}$; on $K_e^{t_1}$,

$$\chi(1+y) = \chi_x(y)$$

for some $x \in A_e^{-t_1}$ (where x is defined mod $A_e^{1-t_1}$). Write

$$x = \alpha_{-t_1}\varpi^{-t_1} + \cdots, \qquad \varpi = \varpi_e;$$

the terms after the first are arbitrary. We require that $\alpha_{-t_1}\varpi^{-t_1}$ generates a field over F of ramification index e_1 and residue class degree f_1. (Condition (1.3)(iv) says that if $\alpha_{-t_1}\varpi^{-t_1}$ generates a field $E_{(t_1)}$, then $e(E_{(t_1)}/F) = e_1$.) We then show that the elements of G commuting with $\chi|_{E_{(t_1)}}$ are those of the form $g = w_1 x w_1'$, where w_1, $w_1' \in K_e^1$ and x commutes with $E_{(t_1)}$. We also begin to define H by declaring that $H^{t_1'} = K_e^{t_1'}$, where $t_1' = [t_1/2] + 1$.

Assume now that $t_2 \geq t_1'$. We show that χ has an extension χ_0 to $K_e^{t_2}$ such than any element of G commuting with $E_{(t_1)}$ commutes with χ_0. (We say that w commutes with χ if $\chi(wyw^{-1}) = \chi(y)$ whenever both are defined.) Using this, we construct extensions of χ to $K_e^{t_2}$ (for now, we denote a typical one by χ_1) with the property that there exists a field E_1 of ramification index e_1 and residue class degree f_1 such that $g \in G$ commutes with χ_1 iff $w = w_1 x w_1'$, where w_1, $w_1' \in K_e^{t_1-t_2+1}$ and x commutes with E_1. The extension of χ to $K_e^{t_2}$ that we want is, however, not χ_1. Any extension of χ to $K_e^{t_2}$ agreeing with χ_1 on $K_e^{t_2+1}$ is of the form

$$\chi(1+y) = \chi_1(1+y)\psi \circ \mathrm{Tr}(x_2'y),$$

where $x_2' \in A_e^{-t_2} \pmod{A_e^{1-t_2}}$. Restrict attention to the elements y commuting with E_1. Then there is a unique element x_2 in the elements commuting with E_1 such that for these elements,

$$\psi \circ \mathrm{Tr}(x_2'y) = \psi \circ \mathrm{Tr}(x_2 y).$$

Again, x_2 is defined mod $A_e^{1-t_2}$. (The construction of x_2 from x_2' is essentially the S_α-map of [15].) We require x_2 to be such that $E_1[x_2]$ is a field of ramification index e_2 and residue class degree f_2 over F. It then turns out that there is a field $E_{(t_2)}$, with $e(E_{(t_2)}/F) = e_2$ and $f(E_{(t_2)}/F) = f_2$, such that the elements of G commuting with χ are precisely those of the form $w_1 w_2 x w_2' w_1'$, where w_1, $w_1' \in K_e^{t_1-t_2+1}$, w_2, $w_2' \in K_e^1$, w_2 and w_2' commute with E_1, and x commutes with $E_{(t_2)}$. (This does not define $E_{(t_2)}$ uniquely.) We also set $t_2' = [t_2/2]+1$ and $H^{t_2'} = H^{t_1'}(K_e^{t_2'} \cap G_1)$, where $G_1 =$ subgroup of elements in $\mathrm{GL}_n(F)$ commuting with E_1.

We now iterate this. This is, we show that if $t_3 \geq t_1'$, then χ has an extension χ_0 to $K_e^{t_3}$ such that any $w \in G$ commuting with $E_{(t_2)}$ commutes with χ_0. (If $t_3 < t_1'$, a modified version holds.) We then consider certain extensions χ_1 to $K_e^{t_3}$ with the property that for a field E_2 of ramification index e_2 and residue class degree f_2 over F, every $w \in G$ commuting with E_2 commutes with χ_1. We require that $\chi(1+y) = \chi_1(1+y)\psi \circ \mathrm{Tr}(x_3'y)$ on $K_e^{t_3}$, where x_3' gives rise to an x_3 commuting with E_2 such that $e(E_2[x_3]/F) = e_3$ and $f(E_2[x_3]/F) = f_3$, and so on. We also extend the definition of H; of course, we need to show that the new definition is consistent with what we have already done. (The details are in Theorems 6.1 and 8.1; the induction hypothesis is complicated so that we can carry along a large number of needed facts. In particular, we compute exactly which elements of G commute with $\chi|_{H^j}$ for each j.) In some cases, we eventually extend χ to a subgroup H of $Z_e K_e$ such that if w commutes with χ, then $w \in H$. (In other cases, noted above, we tensor with Heisenberg representations or cupsidals.) This shows that χ induces irreducibly to a supercuspidal π.

The construction is so arranged that at any time, one is dealing with computations involving only some K_e^j/K_e^{j+1}. Another consequence of the construction is that the "$t_1/2$ problem" is eliminated. Set $t_1' = [t_1/2]+1$, as before. Then any character χ of $K_e^{t_1'}/K_e^{t_1+1} \cong A_e^{t_1'}/A_e^{t_1+1}$ can be written as $\chi(1+y) = \psi \circ \mathrm{Tr}(xy)$ for some $x \in A_e^{-t_1}$ (defined modulo $A_e^{1-t_1'}$). This expression for χ is of great help in analyzing, for example, the elements $w \in \mathrm{GL}_n(F)$ commuting with χ. However, if we need to extend χ to a subgroup of K_e properly containing $K_e^{t_1'}$, then χ no longer has such a simple form. Since the analysis given here does not depend on the above sort of expression for χ, the above difficulty is obviated.

This construction of π parallels the construction of irreducibles in [4]. We use counting arguments based on the Matching Theorem to prove that we have constructed all supercuspidals. Thus we need to compute the number of supercuspidal representations and of other discrete series representations with given conductoral exponent, and compare that number with the corresponding number for division algebras. We also compute the formal degrees of the representations π. These latter computations are similar to those in [6]. They serve the purpose of showing that supercuspidals constructed using $Z_e K_e$ and those constructed using $Z_{e^\sim} K_{e^\sim}$, with $e^\sim \neq e$, are distinct.

As mentioned above, the inductive nature of the construction means that there are large numbers of details to verify at each step. Here is a brief description of the main points needing attention. We need to show that when we extend the definition of H, the new definition is consistent with the old; this involves knowledge of the structure of the G_i and in particular of their relation to each other. The necessary material is developed in §§2 and 3. Section 2 also gives some terminology that is used throughout the paper. We also need to show that χ extends at each step, and, as noted above, we need to be able to compute the set of elements x with $\chi^x = \chi$ (on their common domain) at each step in the construction of χ. The basic lemmas for this are given in §§4 and 5. The main part of the construction is done in §§6 –9; the remaining sections deal with such matters as computing formal degrees and proving completeness. The reader may wish to read the rest of this section and §2 first, and then go to §6, referring to results in the preceding sections as necessary.

This construction is surprisingly close to that given in [10] for the tamely ramified case. Then the character χ is always nontrivial on the field $E_{(j)}$, and one can thus associate irreducible supercuspidals with certain characters of extension fields of F. Furthermore, the geometry (or algebra) of the tamely ramified situation is simpler, and one can arrange to have $E_{(t_1)} \subseteq E_{(t_1-1)} \subseteq \cdots$; this greatly simplifies many arguments.

We shall use some further notational conventions in this paper. Fix the sequence $1 = f_1|f_2|\cdots|f$. There is an embedding of k_f, the extension field of degree f over k, in $M_f(k)$ such that k_{f_1} is diagonally embedded as block $f_1 \times f_1$ matrices (with all blocks the same), k_{f_2} is diagonally embedded as block $f_2 \times f_2$ matrices, and so on. Fix such an embedding. Then $(k_f)^e = k_f \times k_f \times \cdots \times k_f$ (e factors) is embedded in $m_e = M_f(k)^e$, and k_f is embedded as the diagonal. For each i, let $m_e^{f_i}$ be the algebra of elements in m_e commuting with k_{f_i}. Then the algebra of elements commuting with $m_e^{f_i}$ is easily seen to be $k_{f_i}^e$. We are, of course, using coset representations for m_e in the case where $\operatorname{char} F = 0$; in this case, the representatives for elements in k_f can be taken to be 0 or roots of unity in (an appropriate embedding of) the ramified field F_f with $[F_f : F] = f$, and the representatives of elements in $m_e^{f_i}$ can be taken to commute with the cyclic group $k_{f_i}^\times$, and hence with F_{f_i}. (In this paper, F_{f_i} always denotes the unramified extension of F of degree f_i. The elements of $m_e^{f_i}$ are e-tuples of $f \times f$ matrices, each of which is a matrix of $f_i \times f_i$ blocks where each block is an element of k_{f_i}.) Notice further that the representatives for each $(k_{f_i}^\times)^e$ form a finite group, and that those for $(k_{f_i}^\times)^e$ form a subgroup of those for $(k_{f_{i+1}})^e$. We can choose the representatives of m_e so that

(i) they are closed under left multiplication by $(k_f^\times)^e$;

(ii) if α represents an element of m_e, γ represents an element of $k_{f_i}^e$, and $[\alpha, \gamma] \equiv 0 \mod A_e^1$, then $[\alpha, \gamma] = 0$;

(iii) a representative α is invertible iff its image in m_e is invertible.

Here is a proof. It suffices to find a set S of representatives in $M_f(F)$ for $M_f(k)$ satisfying

(i′) S is closed under multiplication by $k_f^\times$;

(ii′) if α represents an element of $M_f(k)$, γ represents an element of k_{f_i}, and $[\alpha, \gamma] \equiv 0 \mod A_e^1$, then $[\alpha, \gamma] = 0$;

(iii′) a representative α is invertible iff its image in $M_f(k)$ is invertible.

(Given such a set, we simply embed $M_f(F)^e$ diagonally in $M_n(F)$ and replicate the representatives.) We can obviously choose representatives satisfying (ii′) and (iii′), with 0 representing 0. The elements of $M_f(k)$ divide into disjoint orbits under multiplication by k_f, and all elements of an orbit commute with $\gamma \in k_{f_i}$ if one element does. For each orbit, choose a single element $\overline{\alpha} \in M_f(k)$, let α be its representative, and redefine the representative of $\overline{\beta\alpha}$ (for $\overline{\beta} \in k_f$) by $\beta\alpha$, where β is the root of unity in F_f corresponding to $\overline{\beta}$. Now (i′)–(iii′) hold. Since we picked the same representatives for each copy of $M_f(k)$, the representatives of m_e are stable under σ.

We generally do not distinguish between elements of m_e and their representatives; the justification will be given in Lemma 3.4, where we show how to go from congruences mod A_e^1 to equalities in $\mathrm{GL}_n(F)$. We also write $m_e^{f_i}(e_i)$, where $e_i|e$, for the elements $\alpha = (\alpha_0, \dots, \alpha_{e-1}) \in m_e^{f_i}$ with $\alpha_{(e/e_i)+j} = \alpha_j \ \ \forall j < e-e/e_i$. For general $\alpha = (\alpha_0, \dots, \alpha_{e-1}) \in m_e$, we define $\mathrm{Tr}_{e_i} \alpha = \sum_{j=0}^{e_i-1} \alpha^{\sigma^{j\,e/e_i}}$; then $\mathrm{Tr}_{e_i} \alpha \in m_e(e_i)$. We set $\mathrm{Tr}\,\alpha = \sum_{j=0}^{e-1} \mathrm{Tr}\,\alpha_i$; however, for $\alpha \in m_e^{f_i}(e_i)$, we set $\mathrm{Tr}^{(e_i)} \alpha = \sum_{j=0}^{(e/e_i)-1} \mathrm{Tr}\,\alpha_j$. Hence for $\alpha \in m_e$, $\mathrm{Tr}^{(e_i)} \mathrm{Tr}_{e_i} \alpha = \mathrm{Tr}\,\alpha$ (in an obvious sense).

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2

In this section we introduce some notation and terminology and give some results on subgroups of $\mathrm{GL}_n(F)$ of the form $\mathrm{GL}_{n_0}(E)$, where E is an extension field of F with ramification index e_0 and residue class degree f_0 (of course, $e_0|e$ and f_0 is one of the f_i), and $e_0 f_0 n_0 = n$. We say that E is *nicely embedded* if the following hold:

(i) $F_{f_0} \subseteq E$. (As in §1, F_{f_0} is embedded "diagonally" in $M_n(F)$.)

(ii) There is a prime element ξ in E such that if we write ξ as an $n/f_0 \times n/f_0$ block matrix (with each block consisting of an entry in F_{f_0}), then the only nonzero blocks are those with indices (i, j) such that $i \equiv j \mod n_0$.

Thus if we rearrange the blocks (numbering the rows and columns from 0 to $n/f_0 - 1$) so that they appear in the order $0, n_0, \dots, (e_0 - 1)n_0, 1, n_0 + 1, \dots, (e_0 - 1)n_0 + 1, \dots, n_0 - 1, \dots, n/f_0 - 1$, then ξ is a "diagonal" matrix of the form $(\xi_0, \dots, \xi_{n_0-1})$, where each ξ_i is an $e_0 \times e_0$ block matrix (whose blocks are elements of F_{f_0}). We require:

(iii) The ξ_i are all equal, and ξ generates $A^1_{e_0}$ over $A^0_{e_0}$.
For example, assume that $n = 6$, $f_0 = f = 1$, $e_0 = 2$. Then (i) is trivial, (ii) says that ξ is of the form

$$\begin{bmatrix} a & 0 & 0 & b & 0 & 0 \\ 0 & c & 0 & 0 & d & 0 \\ 0 & 0 & e & 0 & 0 & f \\ g & 0 & 0 & h & 0 & 0 \\ 0 & i & 0 & 0 & j & 0 \\ 0 & 0 & k & 0 & 0 & l \end{bmatrix} \qquad (g\,,\,i\,,\,k \in P_F\,;\ \text{other entries in } \mathscr{O}_F),$$

and (iii) says that $a = c = e$, $b = d = f$, $g = i = k$, and $h = j = l$. Notice, incidentally, that every element of E is of this form, from our convention about F_{f_0} and the fact that $F_{f_0}[\xi] = E$.

Occasionally we replace (iii) by

(iii′) The ξ_i are conjugate in $\mathrm{GL}_{e_0}(F_{f_0})$ and generate $A^1_{e_0}$ (over $A^0_{e_0}$) there; we then say that E is *embedded*.

Recall that the permutation of rows and columns of a matrix in the same way (so that the ith row becomes the jth and the ith column also becomes the jth) is achieved by a conjugation, $x \mapsto P^{-1}xP$, where P is the permutation matrix with 1's in the entries labeled $(i\,,\,j)$ and 0's elsewhere. (A similar result applies to block matrices.) Let P be the permutation implementing the above permutation of rows and columns. Now let $P^{-1}\eta_* P$ be the $n_0 \times n_0$ block matrix (with $e_0 f_0 \times e_0 f_0$ blocks) such that the $(i\,,\,j)$ block is 0 unless $j-1 \equiv 1 \bmod n_0$, is I if $j-i = 1$, and is ξ_0 if $j = 0$ and $i = n_0 - 1$. Then $(P^{-1}\eta_* P)^{n_0} = P^{-1}\xi P$, so that $\eta_*^{n_0} = \xi$; moreover, η_* is an $n/f_0 \times n/f_0$ block matrix (with block entries in F_{f_0}) such that the only nonzero blocks are those satisfying $j - i \equiv 1 \bmod n_0$. Thus $\eta_0 = \eta_*^{f/f_0}$ generates A^1_e over A^0_e and satisfies $\eta_0^{e/e_0} = \xi$ and $\eta_0 \alpha \eta_0^{-1} = \alpha^\sigma$ if $\alpha \in m_e^{f_0}(e_0)$. In the example above, the permutation matrix P has the effect of shifting the order of rows and columns (originally from 0 to 5) to 0, 3, 1, 4, 2, 5, so that

$$P^{-1}\xi P = \begin{bmatrix} a & b & 0 & 0 & 0 & 0 \\ g & h & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & g & h & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & g & h \end{bmatrix}.$$

Then

$$P^{-1}\eta_* P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ a & b & 0 & 0 & 0 & 0 \\ g & h & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\eta_* = \eta_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ a & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ g & 0 & 0 & h & 0 & 0 \end{bmatrix}.$$

Then matrix algebra $M_{n_0}(E)$ is generated over E by η_* and $m^{f_0}_{n/f_0}(e_0)$ or by η_*^d and $m^{f_0}_{n/df_0}(e_0)$ for any d dividing n/f_0—in particular, by η_0 and $m^{f_0}_e(e_0)$. When we say that E is nicely embedded, we shall also assume that η_0 is of the above form. We also say that $M_{n_0}(E)$ or $\mathrm{GL}_{n_0}(E)$ is nicely embedded (or embedded) if E is.

Call Π and (e_0, f_0)-*permutation matrix* if it is an $e_0 \times e_0$ block matrix where all off-diagonal blocks are 0, the diagonal are equal, and the common diagonal $n_0 f_0 \times n_0 f_0$ block is an $n_0 \times n_0$ permutation block matrix (so that each $f_0 \times f_0$ block is I or 0). Thus if $f_0 = 1$, then Π is a permutation matrix in which the first n_0 rows and columns are permuted, the next n_0 rows and columns are permuted in the same way, and so on. Notice that if $f_0 | f_0^{\sim}$, then every $(e_0, f_0^{\sim})$-permutation matrix is also an (e_0, f_0)-permutation matrix; a similar statement holds if $e_0 | e_0^{\sim}$. The (e_0, f_0)-permutation matrices are all in $\mathrm{GL}_{n_0}(E)$.

Say that an element of $G = \mathrm{GL}_n(F)$ is a *power-permutation matrix* if each row and each column has only one nonzero entry and if that entry is of the form $a\varpi_F^j$, $j \in \mathbb{Z}$ and $a \in k^\times$. Such matrices are products Πu, where Π is a permutation matrix and $u = \mathrm{diag}(a_0\varpi_F^{j_0}, \dots, a_{n-1}\varpi_F^{j_{n-1}})$. Here is an expression for these matrices in the form $\sum_{j=c_0}^{\infty} \alpha_j \varpi^j$, $\alpha_j \in m_e$. Let $b_{i,j,h}$ $(0 \le i, j \le f-1; 0 \le h \le e-1)$ be the element $(b_0, b_1, \dots, b_{e-1})$, where $b_{h'} = 0$ unless $h' = h$ and the only nonzero entry of b_h is a 1 in the (i, j) position. (As above, we label the rows and columns of the b_h from 0 to $e-1$, and those of elements of G from 0 to $n-1$; extend the notation cyclically, so that, e.g., the j_1th row and j_2th row of b_h are the same if $j_1 = j_2 \bmod e$.) The $b_{i,j,h}$ form, of course, the "obvious" basis for m_e. A straightforward computation shows that a power-permutation matrix is of the form $\sum_{l=1}^n a_l b_{i_l, j_l, h_l}\varpi^{m_l}$, where the $a_l \in k^\times$ and the sets $\{fh_l + i_l\}$, $\{f(h_l + m_l) + j_l\}$ $(1 \le l \le n)$ run through the conjugacy classes mod n.

Now consider the group $G_0 = \mathrm{GL}_{n_0}(E)$, where E is a nicely embedded extension field, so that $M_0 = M_{n_0}(E)$ is generated by $m_e^{f_0}(e_0)$ and the element η_0 constructed above. Let $\xi = \eta_0^{e/e_0}$, so that ξ is central in G_0. Recall that $P^{-1}\xi P = \{\xi_0, \dots, \xi_0\}$, a "diagonal" block $n_0 \times n_0$ matrix with all diagonal entries the same. Say that g is a power-permutation matrix of $G_0 = \mathrm{GL}_{n_0}(E)$ if $P^{-1}gP$ is of the form $Q\{\alpha_0\xi_0^{r_0}, \alpha_1\xi_0^{r_1}, \dots, \alpha_{n_0-1}\xi^{r_{n_0-1}}\}$, where Q is an $n_0 \times n_0$ block permutation matrix, the $\alpha_i \in k_{f_0}$, and the r_i are integers. This is consistent with our definition for $\mathrm{GL}_n(F)$. Another description is as follows: let $b'_{i,j,h}$ $(0 \le i, j \le f/f_0 - 1$, $0 \le h \le e/e_0 - 1)$ be the element $(b_0, \dots, b_{e-1}) \in m_e^{f_0}(e_0)$ such that (a) the b_h are periodic with period e/e_0; (b) for $0 \le h' < e/e_0$, $b_{h'} = 0$ unless $h' = h$; and (c) b_h has only one nonzero block, an I

in the (i,j) block. The $b'_{i,j,h}$ give the "natural" basis for $m_e^{f_0}(e_0)$ as a k_{f_0}-space. We can describe η_0 as an $e_0 \times e_0$ block matrix each of whose blocks is itself a block matrix with $f_0 \times f_0$ blocks; as a result, η_0 is an $n/f_0 \times n/f_0$ block matrix where the blocks are elements of k_{f_0}, and the (i,j) block is nonzero only if $j-1 \equiv f/f_0 \bmod n_0$. Hence the (i,j) block of η_0^c is 0 unless $j-i \equiv cf/f_0 \bmod n_0$. Then the power-permutation matrices of G_0 are the elements of the form $\sum_{l=1}^{n_0} \alpha_l b'_{i_l, j_l, h_l} \eta_0^{m_l}$, where the $\alpha_l \in k_{f_0}^\times$ and the sets $\{fh_l/f_0 + i_l\}$, $\{f(h_l+m_l)/f_0 + j_l\}$ $(1 \le l \le n_0)$ run through the conjugacy classes mod n_0. As usual, $G_0 = (K^1_{n/f_0} \cap G_0) W_0 (K^1_{n/f_0} \cap G_0)$, where $W_0 =$ group of permutation-power matrices of G_0.

An example may help. Let $n=6$, $f=1$, and $e_0 = 2$. One possible choice for η_0 is

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \varpi & 0 & 0 & \varpi & 0 & 0 \\ 0 & \varpi & 0 & 0 & \varpi & 0 \\ 0 & 0 & \varpi & 0 & 0 & \varpi \end{bmatrix}.$$

(Of course, ϖ determines E.) Write $\tau = \varpi^2 + \varpi$, so that

$$\xi = \eta_0^2 = \begin{bmatrix} \varpi & 0 & 0 & \varpi & 0 & 0 \\ 0 & \varpi & 0 & 0 & \varpi & 0 \\ 0 & 0 & \varpi & 0 & 0 & \varpi \\ \varpi^2 & 0 & 0 & \tau & 0 & 0 \\ 0 & \varpi^2 & 0 & 0 & \tau & 0 \\ 0 & 0 & \varpi^2 & 0 & 0 & \tau \end{bmatrix}.$$

Therefore

$$\xi_0 = \begin{bmatrix} \varpi & \varpi \\ \varpi_2 & \tau \end{bmatrix}.$$

One permutation matrix (corresponding to

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and ξ_0^2, ξ_0, ξ_0^0) is

$$\begin{bmatrix} \varpi\tau & 0 & 0 & a & 0 & 0 \\ 0 & 0 & \varpi & 0 & 0 & \varpi \\ 0 & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & b & 0 & 0 \\ 0 & 0 & \varpi_2 & 0 & 0 & \tau \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{where } \begin{bmatrix} \varpi\tau & a \\ a & b \end{bmatrix} = \xi_0^2$$

(thus $a = \varpi(\varpi+\tau)$ and $b = \varpi^3 + \tau^2$). If we had $n=12$, $f=2$, and $e_0 = 2$, the situation would be similar, but each entry would be replaced by a 2×2 matrix (corresponding to some element of k_f).

In this paper, we will often construct E by constructing $M_{n_0}(E)$. The following proposition is what we need:

(2.1) **Proposition.** *Let $e_0|e$ and $f_0|f$. Suppose that $\eta_0 \in \mathrm{GL}_n(F)$ satisfies:*

(i) *η_0 generates A_e^1;*

(ii) *η_0 normalizes $m_e^{f_0}(e_0)$, and conjugation by η_0 is σ there.*

Let M_0 be the subalgebra of $M_n(F)$ generated by η_0 and $m_e^{f_0}(e_0)$, and let E_0 be its center. Write $n_0 = n/e_0 f_0$. Then

(a) *$E_0 = F_{f_0}[\eta_0^{e/e_0}]$, and E_0 is an embedded field;*

(b) *$e(E_0/F) = e_0$ and $f(E_0/F) = f_0$;*

(c) *$[M_0 : E_0] = n_0^2$.*

Proof. The hypothesis implies that η_0 commutes with k_f and hence with F_f. Since $1, \eta_0, \dots, \eta_0^{e-1}$ are linearly independent over F_f (because elements of $F_f \cap A_e^1$ are in A_e^e), $F_f[\eta]$ is a commutative subalgebra of dimension $\geq ef = n$ in $M_n(F)$. Therefore it is maximal and $1, \eta_0, \dots, \eta_0^{e-1}$ form an F_f-basis for $F_f[\eta_0] \subseteq \mathrm{GL}_e(F_f)$. This shows that the characteristic polynomial for $\eta_0 \subseteq \mathrm{GL}_e(F_f)$ is irreducible; hence $F_f[\eta_0]$ is a field. Then $E' = F_{f_0}[\eta_0^{e/e_0}]$ is also a field. Write $[E' : F] = h$; $h \geq e_0 f_0$, since $1, \xi, \xi^2, \dots, \xi^{e_0-1}$ are obviously linearly independent over F_{f_0}. In fact, $e(E'/F) \geq e_0$ and $f(E'/F) \geq f_0$.

Clearly k_{f_0} (hence F_{f_0}) and $\xi = \eta_0^{e/e_0}$ commute with $m_e^{f_0}(e_0)$; hence E' commutes with $m_e^{f_0}(e_0)$. We also see immediately that E' commutes with η_0. But the algebra generated by η_0 and $m_e^{f_0}(e_0)$ is an E'-algebra of dimension $\geq n_0^2$ over E'. Let $M^\sim$ be the commutant of E'; $M^\sim \cong M_{n/h}[E']$. Since $M_0 \subseteq M^\sim$ (because η_0 and $m_e^{f_0}(e_0)$ commute with E'), these inequalities imply that $h = e_0 f_0$ and that $M_0 = M^\sim$. Thus we have $e(E'/F) = e_0$ and $f(E'/F) = f_0$. Because ξ commutes with M_0, it must satisfy (i) and (ii) of the criterion for E' to be embedded; because it satisfies an equation of degree e_0 over F_{f_0}, the ξ_i in (iii$'$) must be conjugate in $\mathrm{GL}_{n_0}(F_{f_0})$. $\square$

Let E_0 be nicely embedded, as above, and assume that $f_0 = f$. Let $\eta_0 \in \mathrm{GL}_{n_0}(E_0)$ generate $A_{e_0}^1$, normalize $m_e^{e_0}(f_0)$, and act as σ there. Say that $y = 1 + y_0 \in G$ is (e_0, f_0)-*pure* if there are integers c, l such that y_0 can be written as $\gamma\eta_0^l$ with $\gamma = (\gamma_0, \dots, \gamma_{e-1}) \in m_e$ and the $\gamma_a = 0$ unless $a \equiv c \bmod e/e_0$. Then if one writes y_0 as an $e \times e$ block matrix with $f_0 \times f_0$ blocks, the only nonzero blocks are those with indices (b, b') such that $b \equiv c$ and $b' \equiv c + l \bmod e/e_0$. It is easy to verify that if y is (e_0, f_0)-pure and g is a power-permutation matrix for E, then gyg^{-1} is (e_0, f_0)-pure; furthermore, any element $w \in K_e^1$ is a product of (e_0, f_0)-pure elements (corresponding to distinct pairs (c, l)). We will sometimes need this decomposition of w. Often w will be in a subgroup H; it will always be easy to check when we use the decomposition that the terms y are in H and that if g is a power-permutation matrix for $\mathrm{GL}_{n_0}(E)$, then $gxg^{-1} \in H$ iff $gyg^{-1} \in H$ for every $y^\sim$ in the above factorization.

3

This section contains what might be called approximation lemmas concerning nicely embedded $\mathrm{GL}_{n_h}(E_h)$. Our running assumption for this section is that we have embedded fields $E_1, \dots, E_i$, with $e(E_h/F) = e_h$ and $f(E_h/F) = f_h$; we assume that $1 = e_0|e_1|\cdots|e_i$ and that $1 = f_0|f_1|\cdots|f_i$, but we do not need to know that $e_h f_h > e_{h-1} f_{h-1}$. (In applications, this inequality will hold if $h < i$, but not necessarily for $h = i$.) Set $n_h = n/e_h f_h$. Assume that $M_h = M_{n_h}(E_h)$ is generated by $m_e^{f_h}(e_h)$ and an element η_h such that η_h generates A_e^1, normalizes $m_e^{f_h}(e_h)$, and acts as σ there. We let $E_0 = F$, $M_0 = M_n(F)$, and $\eta_0 = \varpi$.

(3.1) **Lemma.** *Let E_h, M_h, and η_h be as above, and set G_h = group of invertible elements of M_h. Suppose in addition that there are integers $t_1 > t_2 > \cdots > t_i > 0$ such that for $h \geq 2$,*

$$\eta_h = \delta_h \eta_{h-1} + \zeta_{h,h-2} + \cdots + \zeta_{h,1} + \zeta_{h,0},$$

where $\zeta_{h,j} \in A_e^{t_h - t_{j+1}+1} \cap M_j$ and $\delta_h \in F_{f_h}$. Then:

(i) *For every $x_h \in A_e^r \cap M_h$ $(h \geq 1)$, there are elements $x_{h-1} \in M_{h-1}, \ldots, x_0 \in M_0$ such that $x_h = \sum_{j=0}^{h-1} x_j$ and $x_j \in A_e^{r+t_{j+1}-t_h}$, $0 \leq j \leq h-1$.*

(ii) *For every $x_h \in K_e^r \cap G_h$ $(h \geq 1)$, there are elements $y_{h-1} \in G_{h-1}, \ldots, y_0 \in G_0$, such that $x_h = y_{h-1} y_{h-2} \cdots y_0$ and $y_j \in K_e^{r+t_{j+1}-t_h}$, $0 \leq j \leq h-1$.*

(iii) *If $h \geq 1$ and $x_h \in Z_e K_e \cap G_h$ but $x_h \notin K_e^1$, assume that $x_h \in A_e^r$ but $x_h \notin A_e^{r+1}$. Then there are elements $z_{h-1} \in G_{h-1}, \ldots, z_0 \in G_0$ such that $x_h = z_{h-1} z_{h-2} \cdots z_0$, $z_h \in A_e^r \cap G_{h-1}$ but $z_h \notin A_e^{r+1}$, and $z_j \in K_e^{t_{j+1}-t_h}$, $0 \leq j \leq h-2$.*

(iv) *Suppose that r_1, r_2 are integers and $1 \leq h \leq j \leq i$. Then*

$$(A_e^{r_1} \cap M_h)(A_e^{r_2} \cap M_j) \subseteq (A_e^{r_1+r_2} \cap M_{h-1}) + (A_e^{r_1+r_2+t_{h-1}-t_h} \cap M_{h-2}) + \cdots + A_e^{r_1+r_2+t_1-t_h},$$

and similarly for $(A_e^{r_2} \cap M_j)(A_e^{r_1} \cap M_h)$.

(v) *Suppose that r_1, r_2 are integers and that $1 \leq h < j \leq i$. Then*

$$(A_e^{r_1} \cap M_h)(A_e^{r_2} \cap M_j) \subseteq (A_e^{r_1+r_2} \cap M_h) + (A_e^{r_1+r_2+t_h-t_{h+1}} \cap M_{h-1}) + \cdots + A_e^{r_1+r_2+t_1-t_{h+1}},$$

and similarly for $(A_e^{r_2} \cap M_j)(A_e^{r_1} \cap M_h)$.

(vi) *For each $r > 0$ and each $h \leq i$,*

$$\begin{aligned} 1 + A_e^r \cap M_{h-1} + A_e^{r+t_{h-1}-t_h} \cap M_{h-2} + \cdots + A_e^{r+t_1-t_h} \\ = (K_e^r \cap G_{h-1})(K_e^{r+t_{h-1}-t_h} \cap G_{h-2}) \cdots K_e^{r+t_1-t_h} \end{aligned}$$

is a group normalized by $G_h \cap Z_e K_e$; the intersection of this group with G_h is $K_e^r \cap G_h$.

Proof. We use induction on i. For $i = 1$, the only nonvacuous part is the statement in (v) that $Z_e K_e$ normalizes K_e^r, and this is standard. For $i = 2$, it suffices to prove (i) when $x_2 \in A_e^r$ for some $r > 0$ (multiply by a central element if necessary), and it suffices to do this when x_2 is a power of η_2 (any x_2 is a sum of powers of η_2 with coefficients in every G_h). Now (i) is clear because $\eta_2 \equiv u_1 \bmod A_e^{t_1-t_2+1}$ with $u_1 \in G_1$, by hypothesis, and then $\eta_2^r \equiv u_1^r \bmod A_e^{t_1-t_2+r}$. For (ii), write $x_2 = 1 + y_2$, with $y_2 \in A_e^r$, $r > 0$; set $y_2 = y_1 + y_0$, as in (i). Then

$$x_2 = (1 + y_1)(1 + (1 + y_1)^{-1} y_0).$$

Part (iii) is similar: we have $x = z_1 + z_0$, as in (i), and then $x = z_1(1 + z_1^{-1} z_0)$. Part (iv) is trivial unless $h = j = 2$. It then suffices to prove that $\eta_2^{r_1} \eta_2^{r_2} \subseteq A_e^{r_1+r_2} \cap M_1 + A_e^{r_1+r_2+t_1-t_2} \cap M_0$, and this is clear from (i). Part (v) is interesting only if $j = 2$ and $h = 1$. Since $A_e^{r_2} \cap M_2 \subseteq A_e^{r_2} \cap M_2 + A_e^{r_2+t_1-t_2}$, this case is also easy. As for (vi), we need to show first that

$$K_e^{r+t_1-t_2}(K_e^r \cap G_1) = 1 + (A_e^r \cap M_1) + A_e^{r+t_1-t_2}.$$

If $u_0 \in A_e^{r+t_1-t_2}$ and $u_1 \in A_e^r \cap M_1$, then $(1+u_0)(1+u_1) = 1+u_1+(u_0+u_0u_1)$ and $1+u_0+u_1 = (1+u_1)(1+(1+u_1)^{-1}u_0)$, with $u_0+u_0u_1$, $(1+u_1)^{-1}u_0 \in A_e^{r+t_1-t_2}$. This set is clearly a group. That G_2 intersects it in $G_2 \cap K^r$ follows from (i). If $x_2 \in G_2 \cap Z_eK_e$, write $x_2 = z_0z_1$ as in (ii) or (iii). Then z_0 normalizes the group because all commutators with elements of the group lie in $K_e^{r+t_1-t_2}$; z_1 obviously normalizes $K_e^r \cap G_1$, and $K_e^{r+t_1-t_2}$ is normal in Z_eK_e. This finishes the proof of (vi).

Now assume the result for $i-1$; we prove it for i. For (i) it suffices, as in the case $i = 2$, to prove the result for η_i^r when $r \geq 1$. For $r = 1$, it holds by hypothesis. Assume the result for $r-1$. Multiplying out the expansions of η_i^{r-1} and η_i, we see that we need to show that

$$\begin{aligned}&(A_e^{r-1+t_{h+1}-t_i} \cap M_h)(A_e^{1+t_{j+1}-t_i} \cap M_j)\\ &\quad\subseteq (A_e^{r+t_{h+1}-t_i} \cap M_h) + (A_e^{r+t_h-t_i} \cap M_{h-1}) + \cdots + A_e^{r+t_1-t_i}\end{aligned}$$

if $h \leq j$ (or the similar formula if $h > j$). If $h = j$, this is obvious; for $h < j$, this follows from (v). For (iii), write $x = x_{i-1} + \cdots + x_0$, from (i); then

$$x = x_{i-1}(1 + x_{i-1}^{-1}x_{i-2} + \cdots + x_{i-1}^{-1}x_0).$$

Now (v) shows that

$$x_{i-1}^{-1}x_{i-2} + \cdots + x_{i-1}^{-1}x_0 \in (A_e^{t_{i-1}-t_i} \cap M_{i-2}) + \cdots + (A_e^{t_2-t_i} \cap M_1) + A_e^{t_1-t_i}.$$

From (vi), $x_{i-1}^{-1}x \in (K_e^{t_{i-1}-t_i} \cap G_{i-2}) \cdots (K_e^{t_2-t_i} \cap G_1)K_e^{t_1-t_i}$, and (iii) follows. The proof of (ii) is similar, but we use

$$x = 1 + x_{i-1} + \cdots + x_0 = (1 + x_{i-1})(1 + (1 + x_{i-1})^{-1}(x_{i-2} + \cdots + x_0)).$$

Part (iv) follows from the inductive hypothesis unless either $h = i$ or $j = i$. If, say, $h = i > j$, then $A_e^{r_1} \cap M_i \subseteq (A_e^{r_1} \cap M_{i-1}) + (A_e^{r_1+t_{i-1}-t_i} \cap M_{i-2}) + \cdots + A_e^{r_1+t_1-t_i}$, and we again get the result from the inductive hypothesis. So the only case to check is where $h = j = i$. Then $(A_e^{r_1} \cap M_i)(A_e^{r_2} \cap M_i) = (A_e^{r_1+r_2} \cap M_i)$, and the result follows from (i). As for (v), we may again assume by induction that $j = i$. Write $A_e^{r_2} \cap M_i = (A_e^{r_2} \cap M_{i-1}) + (A_e^{r_2+t_{i-1}-t_i} \cap M_i) + \cdots + A_e^{r_2+t_1-t_i}$; then use (v) (with $j < i$) repeatedly and note that $(A_e^{r_1} \cap M_h)(A_e^{r_2+t_{h-1}-t_i} \cap M_h) = A_e^{r_1+r_2+t_{h-1}-t_i} \cap M_h$.

We still need (vi). Set $H_r = (K_e^r \cap G_{i-1})(K_e^{r+t_{i-1}-t_i} \cap G_{i-2}) \cdots K_e^{r+t_1-t_i}$. Since $H_r = \{(Z_eK_e \cap G_{i-1})(K_e^{r+t_{i-1}-t_i} \cap G_{i-2}) \cdots K_e^{r+t_1-t_i}\} \cap K_e^r$, this is a group by (vi) for the case $i-1$. To see that $H_r = 1 + A_e^r \cap M_{i-1} + A_e^{r+t_{i-1}-t_i} \cap M_{i-2} + \cdots + A_e^{r+t_1-t_i}$, write $K_e^r \cap G_{i-1} = 1 + A_e^r \cap M_{i-1}$, etc., and use (v). To show that $G_i \cap Z_eK_e$ normalizes H_r, it suffices to see that η_i and $G_i \cap K_e$ normalize it. To verify that η_i normalizes H_r, note first that $\eta_i \in \eta_j(K_e^{t_j-t_{j+1}} \cap G_{j-1}) \cdots K_e^{t_1-t_{j+1}}$; this follows from repeated use of (v). It is now easy to show that conjugation by η_i maps $K_e^{r+t_{j+1}-t_i} \cap G_j$ into H_r. Since $G_i \cap K_e / G_i \cap K_e^1$ has coset representatives that are also in every G_j with $j < i$, these elements normalize H_r, and we therefore need only prove that $G_i \cap K_e^1$ normalizes H_r. Since $(1+a)y(1+a)^{-1} = y + (ay - ya) - (aya - ya^2) + \cdots$, we see from (v) that this element is in H_r. □

(3.2) **Corollary.** *Let* $H_r = (K_e^r \cap G_{i-1})(K_e^{r+t_{i-1}-t_i} \cap G_{i-2}) \cdots K_e^{r+t_1-t_i}$, *let* $j \geq r$, *and let* h *be the index such that* $t_{h+1} - t_i + r \leq j < t_h - t_i + r$. *(If* $j \geq t_1 - t_i + r$, *then* $h = 0$.*) Then* $H_r \cap K_e^j / H_r \cap K_e^{j+1} \cong G_h \cap K_e^j / G_h \cap K_e^{j+1}$, *and*

$$H_r \cap K_e^j = (K_e^j \cap G_h)(K_e^{r+t_h-t_i} \cap G_{h-1}) \cdots K_e^{r+t_1-t_i}.$$

The subgroups $K_e^{t_1-t_i}$, $K_e^{t_2-t_i} \cap G_1, \ldots, K_e^{t_{i-1}-t_i} \cap G_{i-2}$, $K_e \cap G_{i-1}$ *normalize* H_r, *as does* η_i.

Proof. Any element of H_r is of the form $x = 1 + y_{i-1} + \cdots + y_0$, with $y_l \in M_l \cap A_e^{r+t_{l+1}-t_i}$, from (vi). Suppose that $x \in K_e^j$ as well and that $j \geq t_{i-1} - t_i + r$. Then y_{i-1} is the only term in the sum not automatically in $A_e^{t_{i-1}-t_i+r}$; since $x - 1 \in A_e^{t_{i-1}-t_i+r}$, we must have $y \in A_e^{t_{i-1}-t_i+r}$. From (i), $y \in A_e^{t_{i-1}-t_i+r} \cap G_{i-1} \subseteq A_e^{t_{i-1}-t_i+r} \cap M_{i-2} + \cdots + A_e^{t_1-t_i+r} \cap M_0$. Thus we may delete y_{i-1} from the sum (perhaps changing the other y_h). Proceeding inductively, we see that $x = 1 + y_h + \cdots + y_0$, $y_l \in M_l \cap A_e^{r+t_{l+1}-t_i}$, and $y_h \in A_e^j$ as well. So modulo K_e^{j+1}, $x \equiv 1 + y_h$, $y_h \in M_h \cap A_e^j$ (and y_h determined mod $M_h \cap A_e^{j+1}$). The second part now follows by a proof like that of (ii). For the last part, suppose that $y \in H_r$ and $w \in K^{t_h-t_i} \cap G_{h-1}$, and write $w = 1 + w_0$. (As in Lemma 3.1, there are coset representatives for $K_e \cap G_{i-1} / K_e^1 \cap G_{i-1}$ that obviously normalize H_r, so that we may assume $w \in K_e^1$.) Then $wyw^{-1} = y + (w_0 y - y w_0) - (w_0 y w_0 - y w_0^2) + \cdots$; from (iv) and (v) of the lemma, this expression is in H_r. We saw in the course of proving Lemma 3.1 that η_i normalizes H; since it also normalizes every K_e^r, it normalizes H_r. □

We now prove a similar result for power-permutation matrices. The hypothesis that $f_{i-1} = f_i$ is convenient and does not cause any trouble; it holds in most applications, and in the others one can always work with the compositum of E_{i-1} and F_{f_i}.

(3.3) **Lemma.** *Let notation and hypothesis be as in Lemma* 3.1, *except that the* E_h *are all nicely embedded,* $f_i = f_{i-1}$, *and* $\eta_i \equiv \eta_{i-1} \bmod A_e^2$. *Then for any power-permutation matrix* x *of* G_i, *there is an element* $u \in (K_e^{t_i-t_{i-1}} \cap G_{i-2}) \cdots (K_e^{t_i-t_1} \cap G_0)$ *such that* ux *is a power-permutation matrix of* G_{i-1}.

Proof. All the matrices we will deal with are block matrices with each $f_i \times f_i$ block in F_{f_i}. Hence there is no loss of generality in working only with matrices commuting with F_{f_i}; thus we may (and do) assume that $f_i = 1$. Because of the way we have written permutation matrices, it is more convenient to show that there is an element u such that xu is a power-permutation matrix of G_{i-1}; since the inverse of a power-permutation matrix is a power-permutation matrix, this proves the lemma. (One could also give a direct proof, at the cost of complicating indices.)

Before giving the full proof, we give an example to illustrate the procedure. Suppose that $n = e = 4$, that $e_1 = e_2 = 2$, and that (writing ϖ and ϖ_F)

$$\eta_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \varpi & 0 & 0 & 0 \\ 0 & \varpi & 0 & 0 \end{bmatrix}, \qquad \eta_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \varpi & 0 & \varpi & 0 \\ 0 & \varpi & 0 & \varpi \end{bmatrix}.$$

Then η_1 is composed of blocks ξ_1 and η_2 is composed of blocks ξ_2, where

$$\xi_1 = \begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}, \qquad \xi_2 = \begin{bmatrix} 0 & 1 \\ \varpi & \varpi \end{bmatrix}.$$

Set $a = 1 + \varpi$, $b = 2 + \varpi$, and $c = 1 + 3\varpi + \varpi^2$, and consider the power-permutation matrices for G_1, G_2 corresponding to $\left[\begin{smallmatrix} 0 & \xi_1 \\ \xi_1^4 & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 & \xi_2 \\ \xi_2^4 & 0 \end{smallmatrix}\right]$ respectively. They are

$$x_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \varpi^2 & 0 & 0 & 0 \\ 0 & \varpi & 0 & 0 \\ 0 & 0 & \varpi^2 & 0 \end{bmatrix}, \qquad x_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ a\varpi^2 & 0 & b\varpi^2 & 0 \\ 0 & \varpi & 0 & \varpi \\ b\varpi^3 & 0 & c\varpi^2 & 0 \end{bmatrix}.$$

Notice that $\xi_2^4 = \xi_1^4 \left[\begin{smallmatrix} a & b \\ \varpi b & c \end{smallmatrix}\right]$ and $\xi_2 = \xi_1 \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. From this, $x_2 = x_1 k$, where

$$\begin{bmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 1 \\ \varpi b & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

it should be clear how k is constructed from the two 2×2 matrices above.

We return to the proof. We use the notation of §2 for power-permutation matrices, so that $b'_{r,s,h}$ is the element $(b_0, \dots, b_{e-1})$ in $m_e^1(e_i)$ (recall: $f_i = 1$) such that the $b_{h'}$ are periodic with period e/e_i, $b_{h'} = 0$ unless $h' \equiv h \bmod e/e_i$, and b_h has only one nonzero entry, a 1 in the (r, s) place. Let u_c be a matrix whose only nonzero entries are at indices (r, s), with $r \equiv s \equiv c \bmod n/e_i = n_i$. Then it is easy to check that $b'_{r_l, s_l, h_l} \eta_i^{m_l} u_c = 0$ unless $s_l + f(h_l + m_l) \equiv c \bmod n_i$. Write $a_l = s_l + f(h_l + m_l)$. One consequence of the above calculation is the following: let u be a matrix whose only nonzero entries are at indices (r, s) with $r \equiv s \bmod n_i$, and let $u_{(c)}$ be the matrix whose (r, s) entry is that of u if $r \equiv s \equiv c \bmod n_i$ and is 0 otherwise. Then

$$b'_{r_l, s_l, h_l} \eta_i^{m_l} \cdot u = b'_{r_l, s_l, h_l} \eta_i^{m_l} \cdot u_{a_l}.$$

Let $x = \sum_{l=1}^{n_i} a_l b'_{r_l, s_l, h_l} \eta_i^{m_l}$ and $x' = \sum_{l=1}^{n_i} a_l b'_{r_l, s_l, h_l} \eta_{i-1}^{m_l}$. Write $u(l) = \eta_i^{-m_l} \eta_{i-1}^{m_l}$ and set $u = \sum_{j=1}^{n_i} u(l)_{a_l}$. From the above remarks,

$$x u(l)_{a_l} = a_l b'_{r_l, s_l, h_l} \eta_i^{m_l} \cdot u(l)_{a_l} = a_l b'_{r_l, s_l, h_l} \eta_{i-1}^{m_l};$$

therefore $x' = xu$. Furthermore, the construction gives $u - 1 \in (A_e^{s_i - s_{i-1}} \cap M_{s_{i-2}}) + \cdots + (A_e^{s_i - s_1} \cap M_0)$. This suffices to prove the result, by Lemma 3.1 □

The last result of this section is of a different sort; it lets us replace congruences by equalities in certain circumstances. In it, we assume that the representatives of elements in m_e satisfy the conditions given in §1.

(3.4) **Lemma.** *Let f_0 be one of $f_1, \dots, f_r$, and let e_0, s satisfy $e_0 = e/(e, s)$. Suppose that α is the lift of an element in $k_{f_0}^e$ such that $(\alpha\varpi^s)^{e_0} = \gamma\varpi^{se_0}$, where γ is a root of unity generating F_{f_0}/F. Assume that β is our lifting of $\overline{\beta} \in m_e$ and that $\alpha\beta^{\sigma^s} - \beta\alpha^{\sigma^j} \equiv 0 \bmod A_e^1$. Then $\alpha\varpi^s$ and $\beta\varpi^j$ commute.*

Proof. By hypothesis, $\alpha\varpi^s\beta\varpi^j - \beta\varpi^j\alpha\varpi^s \equiv 0 \bmod A_e^{j+s+1}$. An easy induction gives $(\alpha\varpi^s)^{e_0}\beta\varpi^j - \beta\varpi^j(\alpha\varpi^s)^{e_0} \equiv 0 \bmod A_e^{j+e_0 s+1}$, or (letting $e_0 s = et$)

$\gamma\varpi_F^t\beta\varpi^j - \beta\varpi^j\gamma\varpi_F^t \equiv 0 \mod A_e^{j+e_0s+1}$. Since ϖ_F^t is central and γ commutes with ϖ, this gives $\gamma\beta - \beta\gamma \equiv 0 \mod A_e^1$. Let $\overline{\gamma}$ be the image of γ in $k_{f_0}^e$. Then $\overline{\gamma}\beta - \overline{\beta}\gamma = 0$, or $\beta \in m_e^{f_0}$. Since $\gamma \in F_{f_0}$ and we chose representatives of $m_e^{f_0}$ to commute the representatives of $k_{f_0}^e$, $\gamma\beta = \beta\gamma$ and the components of β commute with those of α. Therefore

$$\alpha\varpi^s\beta\varpi^j = \alpha\beta^{\sigma^s}\varpi^{s+j} = \beta\alpha^{\sigma^j}\varpi^{s+j} = \beta\varpi^j\alpha\varpi^s$$

(both $\alpha\beta^{\sigma^s}$ and $\beta\alpha^{\sigma^j} = \alpha^{\sigma^j}\beta$ are in $m_e^{f_0}$, since $m_e^{f_0}$ is closed under left multiplication by $k_{f_0}^e$ and under σ; as $\alpha\beta^{\sigma^s} \equiv \beta\alpha^{\sigma^j} \mod A_e^1$, we must have equality). $\square$

Remark. We often use this result with nicely embedded fields and matrix algebras, replacing ϖ with η_0.

4

In this section we give results about elements commuting with various characters. The first few are like results in [4] and are used to prove that certain characters can be extended; they involve commutators.

(4.1) **Lemma.** *Fix e, and let $x \in K_e^r \cap (G, G)$, where $r \geq 1$ and $G = \mathrm{GL}_n(F)$. Assume that the residue class field k has more than two elements. Then there are elements u_j and v_j $(1 \leq j \leq s)$ such that each $v_j \in K_e^r$, each u_j is either $\alpha_j\varpi$ or α_j, where $\alpha_j \in m_e^\times$, and*

$$x \equiv (u_1, v_1)\cdots(u_s, v_s) \mod K_e^{r+1}.$$

Proof. Suppose first that $n \nmid fr$, so that $e \nmid r$. We have

$$(\alpha, 1 + \beta\varpi^r) \equiv 1 + (\alpha\beta(\alpha^{\sigma^r})^{-1} - \beta)\varpi^r \mod K_e^{r+1}.$$

Let $S =$ span of the elements $\alpha\beta(\alpha^{\sigma^r})^{-1} - \beta$. Then $x = 1 + \gamma\omega^r$ is $(\mathrm{mod}\, K_e^{r+1})$ a product of these commutators if $\gamma \in S$. We show that $S = m_e$ when card $K > 2$. Suppose that $\delta \perp S$ under the bilinear map $(\gamma, \delta) = \mathrm{Tr}\, \gamma\delta$. Then for all $\alpha \in m_e^\times$ and all $\beta \in m_e$,

$$0 = \mathrm{Tr}\, \delta(\alpha\beta(\alpha^{\sigma^r})^{-1} - \beta).$$

Replacing β by $\beta\alpha^{\sigma^r}$, we get

$$0 = \mathrm{Tr}\, \alpha(\beta\delta - (\delta\beta)^{\sigma^{-r}}).$$

By the hypothesis on k, $m_e^\times$ spans m_e. (If $f > 1$, then the matrix $e_{i,j}$ with one nonzero element, 1 in the (i, j) place, is $(I + e_{i,j}) - I$ if $i \neq j$; if $i = j$, it is $(J + e_{i,i}) - J$, where J is a cyclical permutation matrix. If $f = 1$, we need k to have > 2 elements; then the construction is easy.) Therefore this last holds for all $\alpha \in m_e$, and $\beta\delta = (\delta\beta)^{\sigma^{-r}}$ $\forall \beta \in m_e$. Let $\beta = (\beta_0, \dots, \beta_{e-1})$ have $\beta_j = I$ and $\beta_i = 0$ for $i \neq j$. Then $\delta = (\delta_0, \dots, \delta_{e-1})$ must have $\delta_j = 0$. Hence $\delta = 0$, and $S = m_e$.

Assume $n | fr$; let $fr = nt$, and write $x = 1 + \gamma\varpi^r + \cdots$. A straightforward computation gives

$$\mathrm{Det}\, x \equiv 1 + (\mathrm{Tr}\, \gamma)\varpi_F^t \mod P_F^{t+1}.$$

Hence $\operatorname{Tr}(\gamma)=0$. We also have

$$(\alpha\varpi\,,\,1+\beta\varpi^r)=1+(\alpha\beta^\sigma\alpha^{-1}-\beta)\varpi^r \mod K_e^{r+1}.$$

Thus is suffices to show that the span S of the elements $\alpha\beta^\sigma\alpha^{-1}-\beta$ is the space T of elements γ with $\operatorname{Tr}\gamma=0$. Since $S\subseteq T$, we need only show that $S^\perp\subseteq T^\perp$ under the bilinear form $(\gamma,\delta)=\operatorname{Tr}\gamma\delta$.

So suppose that $0=\operatorname{Tr}(\alpha\beta^\sigma\alpha^{-1}-\beta)\delta=\operatorname{Tr}\beta[(\alpha^{-1}\delta\alpha)^{\sigma^{-1}}-\delta]$ for all $\alpha\in m_e^\times$ and all $\beta\in m_e$. Then $(\alpha^{-1}\delta\alpha)^{\sigma^{-1}}=\delta\ \ \forall\alpha\in m_e^\times$. For $\alpha=I$, we see that $\delta^{\sigma^{-1}}=\delta$, or that $\delta=(\delta_0,\delta_0,\dots,\delta_0)$; it is now also clear that δ_0 must commute with every element of $\mathrm{GL}_e(k)$, so that δ is central. But then $\delta\in T^\perp$, and we are done. □

Remark. An obvious induction lets us write x modulo K_e^s $(s>r)$ as a product of commutators. Lemma 4.1 (like the other results of this section) will also be applied to $x\in K_e^r\cap([Z_eK_e\cap M_{n_0}(E_0)],\,[Z_eK_e\cap M_{n_0}(E_0)])$, where E_0 is nicely embedded.

The next result depends on Lemma 2.2 of [4]. We give a restatement here in a form that will be useful. Let $\mathscr{A}=\mathbb{Z}[[a,b]]$ be the ring of formal power series in noncommuting variables a, b, and regard $\mathscr{A}$ as a subring of $\mathscr{B}=\mathscr{A}[a^{-1},b^{-1}]$. Let $\mathscr{A}_n=$ ideal in $\mathscr{A}$ generated by all words of length n. For r, s positive integers, give each word in $\mathscr{A}$ a weight by giving a weight r and b weight s, and summing the weights of the letters in a word to get the weight of the word (e.g., $abab^2$ has weight $2r+3s$). Consider

$$x=(1+a)(1+b)(1+a)^{-1}(1+b)^{-1}=(1+a)(1+b)(1-a+a^2+\cdots)(1-b+\cdots).$$

Given any integer $n>0$, there exist an integer $N>0$ and elements $c_1,\dots,c_N$, $d_1,\dots,d_N\in\mathscr{A}$ such that:

(i) x is congruent mod $\mathscr{A}_n$ to the product of commutators $(c_1,d_1)\cdots(c_N,d_N)$;

(ii) d_j-1 is a word, c_j is one of a, a^{-1}, b, b^{-1}, and $c_jd_jc_j^{-1}\in\mathscr{A}$;

(iii) if d_j-1 has weight $\le 2s$, then $c_j=a$ or a^{-1}.

(One can replace (iii) by (iii$'$): if d_j-1 has weight $\le 2r$, then $c_j=b$ or b^{-1}. Note that if d_j-1 has weight $2s$, then the proof implies that it cannot be b^2.)

(4.2) **Lemma.** *With the above notation, suppose that E_0 is nicely embedded in $M_n(F)$ and that $\operatorname{card}(k)>2$. Set $M_0=M_{n_0}(E_0)$ and let η_0 be the element of M_0 that generates A_e^1, normalizes $m_e^{e_0}(f_0)=m_0$, and acts on m_0 as σ. Let χ be a character defined on a subgroup H of K_e^1 containing some K_e^s, with χ trivial on K_e^s, and assume that H is generated by a set of elements of the form $1+\beta\eta_i^r$, $\beta\in m_e^\times$, and η_i a generator of A_e^1 over A_e^0. Assume also that $\mathrm{GL}_{n_0}(E_0)\cap Z_eK_e$ normalizes H and that $\{x-1 : x\in H\}$ is closed under multiplication and under multiplication by $m_0^\times$ and by η_0.*

(a) *If $\chi^w=\chi$ for all $w\in m_0^\times$ and for $w=\eta_0$, then $\chi^w=\chi$ for any $w\in\mathrm{GL}_{n_0}(E_0)\cap Z_eK_e$.*

(b) *If*

(i) $H\subseteq K_e^r$;

(ii) *for all* $j \geq r$ *the group* $H \cap K_e^j / H \cap K_e^{j+1}$ *has a set of (coset representatives of) generators* $y = 1 + y_0$ *such that for all such* y, y_0 *is invertible and normalizes* χ;

(iii) $\chi^{y_0} = \chi$; *and*

(iv) $\chi^w(y) = \chi(y)$ *for all* $y \in H \cap K_e^{r+2t}$ *when* $w \in m_0^\times$ *or* $w = \eta_0$,

then $\chi^w = \chi$ *(on* H*) for all* $w \in \mathrm{GL}_{n_0}(E_0) \cap K_e^t$.

Proof. (a) Any $w \in \mathrm{GL}_{n_0}(E_0) \cap Z_e K_e$ can be written as

$$w = \alpha_0 \eta_0^{j_0}(1 + \alpha_1 \eta_0 + \alpha_2 \eta_0^2 + \cdots), \qquad \alpha_j \in m_0 \text{ and } \alpha_0 \in m_0^\times .$$

Since $\alpha_0 \eta_0^{j_0}$ fixes χ, we may assume that $\alpha_0 = 1$ and $j_0 = 0$. Furthermore, an easy induction makes it clear that is suffices to consider monomials $w = 1 + \alpha \eta_0^j$, $j > 0$. Similarly, it suffices to consider $\chi^w(y)\chi(y^{-1})$ for generating elements $y = 1 + \beta \eta_i^r \in H$. If α is invertible, the result holds by an application of Lemma 2.2 in [4]. If $\alpha = \beta + \gamma$, where β, $\gamma \in m_0^\times$, then it holds because we have

$$1 + \alpha \varpi_0^j = (1 + \beta \varpi_0^j)(1 + \gamma \varpi_0^j)(1 - \beta \varpi_0^j \gamma \varpi_0^j) \cdots ,$$

where for any s the formula holds mod K_e^s after finitely many terms. Since the result holds for each term on the right (by Lemma 2.2 of [4]), it holds for $\alpha \varpi_0^j$. A similar calculation shows that a similar result holds if α is any linear combination of invertible elements. But if k has more than two elements, then every element in m_0 is a linear combination of invertible elements.

(b) The proof is essentially the same. The main point to notice is that if $w = 1 + w_0 \in \mathrm{GL}_{n_0}(E_0) \cap K_e^t$ and $1 + y_0 \in H$, then words in w_0 and y_0 that appear in the commutators and have only one w_0 can be taken care of (according to the result given before the lemma) by conjugating by y_0, and $\chi^{y_0} = \chi$. Those with at least two w_0's are all in $H \cap K_e^{r+2t}$, and there $\chi^{w_0} = \chi$ as well. □

(4.3) *Note.* We often can apply the reasoning of Lemma 4.2 even when the hypotheses do not apply. Suppose that we have a subgroup H of K_e^r, a character χ on H, an element $1 + y \in H$, $y = \beta \eta_0^r$, and an element $1 + \alpha \eta_0^j = 1 + (\alpha_1 + \cdots + \alpha_h)\eta_0^j$ such that:

(i) $\alpha_i \in m_e^\times$, all i, and $\beta \in m_e^\times$;

(ii) if w is any sum of words of the form $u_1 u_2 \cdots u_m$, where $m \geq 2$ and each u_j is either y or an $\alpha_i \eta_0^j$, then $1 + w \in H$;

(iii) $\chi^x = \chi$ (in that they agree on their common domain) if $x = \alpha_i \eta_0^j$;

(iv) k has more than two elements.

Then the reasoning of Lemma 4.2 shows that $\chi((1 + \alpha \eta_0^j, 1 + y)) = 1$. When we use this reasoning we refer to (4.3). It should be clear when we so refer that the conditions are met. A similar argument shows that β need only be a sum of invertible elements.

(4.4) *Remark.* The restrictions on k in (4.1)–(4.3) are annoying. In our uses of these results, it is possible to avoid the restrictions. Consider, for example, Lemma 4.2 when $q = 2$. Let N be a large prime (larger than n), and let $F^\sim$ be the unramified extension of F with $[F^\sim : F] = N$. Choose a character $\psi^\sim$ of $F^\sim$ so that $\psi^\sim|_F = \psi$. Then G embeds naturally into $G^\sim = \mathrm{GL}_n(F^\sim)$, and we have subgroups $Z_e^\sim$, $K_e^{\sim r} \subset G^\sim$ defined like Z_e, $K_e^r \subset G$. In our constructions of χ and H, there will always be a corresponding character $\chi^\sim$ on a subgroup $H^\sim \subset K_e^\sim$ such that $H^\sim = H \cap G$ and $\chi^\sim|_H = \chi$; furthermore, the hypotheses of Lemma 4.2 will also hold for $(m_e^\sim)^f(e_0)^\times$, the group in $K_e^\sim$ corresponding to $m_e^f(e_0)^\times \subset K_e$. Let $E_0^\sim$ be the compositum of E_0 and $F^\sim$. Lemma 4.2 says that if $w \in \mathrm{GL}_{n_0}(E_0^\sim) \cap Z_e^\sim K_e^\sim$, then $(\chi^\sim)^w = \chi^\sim$. For $w \in \mathrm{GL}_{n_0}(E) \cap Z_e K_e = \mathrm{GL}_{n_0}(E_0^\sim) \cap Z_e^\sim K_e^\sim \cap G$, we then see that w normalizes $H^\sim \cap G = H$ and $\chi^w = \chi$. This means that in all cases where we will apply Lemma 4.2, it will hold even if $q = 2$. The same will apply to our uses of (4.3). When we use Lemma 4.1 to show that certain characters extend (because they are trivial on commutators), a similar argument extends the results to the case where $q = 2$. In the rest of this paper, we will apply (4.1)–(4.3) to the case $q = 2$ without comment. A similar remark applies to the following lemma.

(4.5) **Lemma.** *With notation as above, assume that k has more than two elements, and let χ be the character on K_e^j, trivial on K_e^{r+1} $(r \geq j > 0)$, such that $Z_e K_e$ commutes with χ. Then χ factors through* Det *(and hence extends to G as a character).*

Proof. It suffices to show that $\chi(y) = 1$ if $y \in K_e^j$ and $\mathrm{Det}\, y = 1$. Then $y \in (G, G)$, and Lemma 4.1 shows that y is $(\mathrm{mod}\, K_e^{r+1})$ a product of commutators $uvu^{-1}v^{-1}$ with $\chi(uvu^{-1}v^{-1}) = 1$. □

(4.6) **Lemma.** *Let χ be a character on K_e^m, $m \geq 1$, that is trivial on K_e^{m+1}. Let α be such that $\chi(1+\gamma\varpi^m) = \psi \circ \mathrm{Tr}(\alpha^{\sigma^m}\gamma) = \psi \circ \mathrm{Tr}(\alpha\gamma^{\sigma^{-m}})$, $\forall \gamma \in m_e$, and assume that $\alpha\varpi^{-m}$ generates a nicely embedded field E_0 of ramification index e_0 and residue class degree f_0 and that $\alpha \in k_{f_0}^e$. Write $M_0 =$ ring of elements commuting with $\alpha\varpi^{-m}$, $G_0 = G \cap M_0$. Let $j < m$, and let e_1, f_1 satisfy $e_0|e_1$, $f_0|f_1$, $e|je_1$. For $\delta \in k_{f_1}^e$, set*

$$\chi_\delta(y) = \chi(wyw^{-1}y^{-1}),$$

where $y = 1 + \gamma\varpi^j$ and $w = 1 + \delta\varpi^{m-j}$. Then:

(i) *$\chi_\delta(y) = 1$ for all δ iff $y \in K_e^j \cap G_0$.*

(ii) *χ_δ is trivial iff $w \in G_0$.*

(iii) *The χ_δ exhaust the characters $\chi^\#$ on K_e^j trivial on $K_e^{j+1}(K_e^j \cap G_0)$ and of the form $\chi^\#(1+\gamma\varpi^j) = \psi \circ \mathrm{Tr}(\varepsilon\gamma^{\sigma^{-j}})$ for some $\varepsilon \in k_{f_1}^e$.*

Proof. Consider the χ_δ, δ as above. These δ obviously form a k_{f_1}-vector space of dimension e; furthermore, we have

$$\begin{aligned}\chi(wyw^{-1}y^{-1}) &= \chi(1 + (\gamma\delta^{\sigma^j} - \delta\gamma^{\sigma^{m-j}})\varpi^m) \\ &= \psi \circ \mathrm{Tr}\,\alpha(\gamma\delta^{\sigma^j} - \delta\gamma^{\sigma^{m-j}})^{\sigma^{-m}} = \psi \circ \mathrm{Tr}\,\gamma(\delta\alpha^{\sigma^{m-j}} - \alpha\delta^{\sigma^{-m}})^{\sigma^j}.\end{aligned}$$

This is identically 1 for all γ iff $[\delta\varpi^{m-j}, \alpha\varpi^{-m}] \equiv 0 \bmod A_e^{1-j}$. As we saw in Lemma 3.4, this means that $\delta\varpi^{m-j}$ commutes with $\alpha\varpi^{-m}$, which proves (ii). (Notice that $F[(\alpha\varpi^{-m})^{e_0}] = F_{f_0}$, so that the hypotheses of Lemma 3.4 are satisfied.) To see that the subspace of such δ has dimension e/e_0 over k_{f_1}, written $\delta = (\delta_0, \dots, \delta_{e-1})$, $\alpha = (\alpha_0, \dots, \alpha_{e-1})$. Then $\delta\alpha^{\sigma^{m-j}} = \alpha\delta^{\sigma^{-m}}$, so that $\delta_i\alpha_{i-j+m} = \delta_{i-m}\alpha_i$ for all i (all entries commute; we extend the definition of the δ_i and α_i cyclically). Since the α_i are nonzero, all δ_l with $l \equiv i \bmod m$ are determined by δ_i. Therefore the subspace of these δ has dimension (e, m). But $e/(e, j) = e_0$.

The characters χ_δ are all trivial on elements $y \in K_e^j \cap G_0$, since

$$\chi(wyw^{-1}y^{-1}) = \psi \circ \operatorname{Tr} \delta(\alpha\gamma^{\sigma^{-m}} - \gamma\alpha^{\sigma^j})^{\sigma^{m-j}},$$

and the term in parentheses is 0 iff $[\gamma\varpi^j, \alpha\varpi^{-m}] = 0$ (see Lemma 3.4). This proves (i).

If $\beta \in (m_e^{f_1}(e_1))^\times$, then β commutes with ϖ^m and with χ, so that

$$\begin{aligned}\chi_\delta^\beta(y) &= \chi(w\beta y\beta^{-1}w^{-1}\beta y\beta^{-1}) = \chi(\beta wyw^{-1}y^{-1}\beta^{-1}) \\ &= \chi^\beta(wyw^{-1}y^{-1}) = \chi(y).\end{aligned}$$

Hence every χ_j commutes with $(m_e^{f_1}(e_1))^\times$ and is therefore a $\chi^\#$.

The characters $\chi^\#$ satisfy

$$\chi^\#(y) = \chi^\#(1 + \gamma\varpi^j) = \psi \circ \operatorname{Tr}(\varepsilon\gamma^{\sigma^j})$$

for some $\varepsilon \in K_{f_1}^e$. Since $\psi^\#(y) = 1$ for $\gamma\varpi^j \in M_0$, we have $\psi \circ \operatorname{Tr}(\varepsilon\gamma^{\sigma^j}) = 1$ if the entries of γ are periodic with period e_0, or $\psi \circ \operatorname{Tr}^{(e_0)}(\gamma \operatorname{Tr}_{e_0} \varepsilon) = 1$ for all such γ. Therefore $\operatorname{Tr} e_0(\varepsilon) = 0$. Thus the ε in question form a $k_{f_1}^e$-space of dimension $e - e/e_0$. Hence every $\chi^\#$ is a χ_δ, and (iii) is verified. □

Essentially the same proof used to demonstrate (i) and (ii) above also proves:

(4.7) **Lemma.** *Let χ be a character on K_e^m trivial on K_e^{m+1} and given on elements $y = 1+\gamma\varpi^m$ $(\gamma \in m_e)$ by $\chi(y) = \psi\circ\operatorname{Tr}(\alpha^{\sigma^m}\gamma)$, where $E_0 = F[\alpha\varpi^{-m}]$ is a nicely embedded field with $e(E_0/F) = e_0$, $f(E_0/F) = f_0$, and $\alpha \in k_{f_0}^e$. Define G_0 = subgroup of elements commuting with $\alpha\varpi^{-m}$. Fix j, $1 \le j < m$. For $\delta \in m_e$, define χ_δ on K_e^j by*

$$\chi_\delta(y) = \chi(wyw^{-1}y^{-1}),$$

where $y = 1+\gamma\varpi^j$ and $w = 1+\delta\varpi^{m-j}$. (Hence χ_δ is trivial on K_e^{j+1}.) Then:

(i) *$\chi_\delta(y) = 1$ for all δ iff $y \in K_e^j \cap G_0$.*
(ii) *χ_δ is trivial iff $w \in G_0$ (i.e., $\delta\varpi^{m-j}$ commutes with E_0).* □

(4.8) **Lemma.** *Let χ, E_0, and G_0 be as in Lemma 4.7, but define χ_δ (for $\delta \in m_e^\times$) by $\chi_\delta(y) = \chi(wyw^{-1}y^{-1})$, $w = \delta\varpi^j$. Then:*

(i) *$\chi_\delta(y) = 1$ for all δ iff $y \in K_e^m \cap G_0$.*
(ii) *χ_δ is trivial iff $w \in G_0$.*

Proof. We have

$$\begin{aligned}\chi_\delta(1 + \gamma\varpi^m) &= \psi \circ \operatorname{Tr} \alpha^{\sigma^m}(\delta\gamma^{\sigma^j}(\delta^{-1})^{\sigma^m} - \gamma) \\ &= \varphi \circ \operatorname{Tr} \gamma(\alpha^{\sigma^{m-j}}\delta^{\sigma^{-j}}(\delta^{-1})^{\sigma^{m-j}} - \alpha^{\sigma^m}).\end{aligned}$$

This is 0 for all γ iff $\alpha R^{\sigma^{m-j}}\delta^{\sigma^{-j}}(\delta^{-1})^{\sigma^{m-j}} - \alpha^{\sigma^m} = 0$, or iff $(\alpha\delta^{\sigma^{-m}})^{\sigma^{m-j}} = (\alpha^{\sigma^j}\delta)^{\sigma^{m-j}}$, or iff $\alpha\varpi^{-m}$ and $\delta\varpi^j$ commute (this uses Lemma 3.4). The other half is similar. □

5

This section is concerned with results used to show that if $\chi^x = \chi$, then the choice of x is restricted in some way. The first two lemmas concern conjugacy. Suppose that $x = \alpha\varpi^m$ and $x^\sim = \beta\varpi^m$, α, $\beta \in (k_{f_i}^\times)^e$ (for some f_i), are known to generate nicely embedded fields over F and to be conjugate in $\mathrm{GL}_n(F)$. We will need to know that they are conjugate in $Z_e K_e$. If the fields that they generate have ramification index e_0, then e_0 is the smallest positive integer such that $e|e_0 m$; $x^{e_0} = \gamma\varpi^{e_0 m}$, where γ generates an unramified extension over F. A similar statement applies to $(x^\sim)^{e_0} = \delta\varpi^{e_0 m}$. Furthermore, γ and δ are conjugate in $m_e^\times$. If we write $\alpha = (\alpha_0, \ldots, \alpha_{e-1})$, etc.; then $\gamma_0 = \alpha_0\alpha_m \cdots \alpha_{m(e_0-1)}$ (where we extend the definition of the α_i by making them periodic mod e), etc., that is, γ is a sort of norm. Furthermore, $\varepsilon^{-1}\alpha\varpi^m\varepsilon = (\varepsilon^{-1}\alpha\varepsilon^{\sigma^m})\varpi^m$. It is not hard to reduce questions about conjugacy of these elements to the case $m = 1$; then the following lemmas give what we need.

For $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{e-1}) \in m_e$, define

$$N_e(\alpha) = \alpha\alpha^\sigma \cdots \alpha^{\sigma^{e-1}} = (\alpha_0\alpha_1 \cdots \alpha_{e-1}, \alpha_1\alpha_2 \cdots \alpha_{e-1}\alpha_0, \ldots, \alpha_{e-1}\alpha_0 \cdots \alpha_{e-2}).$$

Note that the entries in $N_e(\alpha)$ have the same determinant, but need not be equal unless $e = 1$. Similarly, we define $N_{e_i}\alpha = \alpha\alpha^{\sigma^{e/e_i}}\alpha^{\sigma^{2e/e_i}} \cdots \alpha^{\sigma^{(e_i-1)e/e_i}}$; thus $\gamma = N_{e_0}(\alpha)$ above if $m = e/e_0$.

(5.1) **Lemma.** *Let α, $\beta \in (m_e)^\times$; set $\gamma = N_e(\alpha)$, $\delta = N_e(\beta)$, and write $\gamma = (\gamma_0, \ldots, \gamma_{e-1})$, etc. Suppose that γ_0, δ_0 are conjugate elements in $\mathrm{GL}_e(k)$. Then there exists $\varepsilon \in m_e^\times$ such that $\varepsilon^{-1}\alpha\varepsilon^\sigma \equiv \beta \bmod A_e^1$. If α, $\beta \in (k_e^\times)^f$, then there exists $\varepsilon \in K_e$ with $\varepsilon^{-1}\alpha\varepsilon^\sigma = \beta$.*

Proof. Let $\gamma = (\gamma_0, \ldots, \gamma_{e-1})$ and $\delta = (\delta_0, \ldots, \delta_{e-1})$. We show first that the lemma holds if $\gamma_0 = \delta_0$. Set $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_{e-1})$; evidently we need to satisfy

$$\begin{gathered}
\varepsilon_0^{-1}\alpha_0\varepsilon_1 \equiv \beta_1 \\
\varepsilon_1^{-1}\alpha_1\varepsilon_2 \equiv \beta_2 \\
\vdots \\
\varepsilon_{e-1}^{-1}\alpha_{e-1}\varepsilon_0 \equiv \beta_{e-1}.
\end{gathered}$$

Set $\varepsilon_0 = I$. Then the first $(e-1)$ equations give

$$\begin{aligned}
\varepsilon_1 &\equiv \alpha_0^{-1}\beta_0, \\
\varepsilon_2 &\equiv \alpha_1^{-1}\alpha_0^{-1}\beta_0\beta_1, \\
&\vdots \\
\varepsilon_{e-1} &\equiv \alpha_{e-2}^{-1} \cdots \alpha_1^{-1}\alpha_0^{-1}\beta_0\beta_1 \cdots \beta_{e-2}.
\end{aligned}$$

But now the last equation holds as well, since it amounts to the statement that $\gamma_0 = \delta_0$.

In general, we know that $\delta_0 \equiv \zeta_0^{-1}\gamma_0\zeta_0^\sigma$ for some matrix $\zeta_0 \in \mathrm{GL}_f(k)$. Set $\zeta = (\zeta_0, \zeta_0, \ldots, \zeta_0)$. Then $\zeta^{-1}\alpha\zeta$ and β are σ-conjugate by the first part of the proof.

If α, $\beta \in (k_f^\times)^e$, then there is a $\zeta_0 \in \mathrm{GL}_f(\mathscr{O}_F)$ such that conjugating by ζ_0 is an automorphism of F_f taking γ_0 to δ_0. We use $\zeta = (\zeta_0, \zeta_0, \dots, \zeta_0)$ to reduce to the case where $\gamma_0 = \delta_0$. The rest of the proof now goes as before, but the congruences can be replaced by equalities because all elements α_i, β_i, ε_i are in k_f. □

(5.2) **Lemma.** *Suppose that $\alpha\varpi_e^j$ and $\beta\varpi_e^j$ (with α, $\beta \in (k_f^\times)^e$) generate fields over F and are conjugate in G. Then they are conjugate in K_e.*

Proof. Let $(j, e) = r$, $e = re_1$, so that $F[\alpha\varpi_e^j]$ has ramification index e_1 over F; let the residue class degree be f_1. Let $(\alpha\varpi_e^j)^{e_1} = \gamma\varpi_e^{je_1}$ and $(\beta\varpi_e^j)^{e_1} = \delta\varpi_e^{je_1}$. We have $\gamma = (\gamma_0, \dots, \gamma_{e-1})$ and $\delta = (\delta_0, \dots, \delta_{e-1})$, where the γ_i, δ_i are all conjugate elements of k_{f_1}. Thus we may (by conjugating) assume that $\gamma = \delta$. Since $\gamma = N_{e_1}\alpha$, $\delta = N_{e_1}\beta$, and conjugating, e.g., $\alpha\varpi^j$ by $\varepsilon^{-1} \in m_e^{f_i}(e_i)$ gives $\varepsilon^{-1}\alpha\varepsilon^{\sigma^j}\varpi^j$ (where σ^j and σ^{e_1} generate the same group), Lemma 5.1 (applied with σ^{e_1}, to the elements α_j with j in a fixed congruence class mod e_1) gives the result. □

We use these lemmas for a further result about conjugacy which we need later. We will have characters χ, $\chi^\sim$ defined on K_e^m $(m \geq 1)$, trivial on K_e^{m+1}, and given on K_e^m by

$$\chi(1+y) = \psi \circ \mathrm{Tr}(xy), \qquad \chi^\sim(1+y) = \psi \circ \mathrm{Tr}(x^\sim y),$$

with $x = \alpha\varpi^{-m}$, $x^\sim = \alpha^\sim\varpi^{-m}$, and α, $\alpha^\sim \in m_e^\times$.

(5.3) **Proposition.** *Use the above notation. Suppose that x, $x^\sim$ generate fields over F and that for some $w \in G$, $\chi(wuw^{-1}) = \chi^\sim(u)$ for all $u \in K_e^m \cap w^{-1}K_e^m w$. Then x, $x^\sim$ are conjugate in Z_eK_e.*

Proof (adapted from [10]). By assumption, $\chi^w\chi^{\sim-1}$ is trivial on $K_e^m \cap w^{-1}K_e^m w$ and is given by $\chi^w\chi^{\sim-1}(1+y) = \psi \circ \mathrm{Tr}((w^{-1}xw - x^\sim)y)$ on K_e^m. Therefore $w^{-1}xw - x^\sim \in (A_e^m \cap w^{-1}A_e w)^\perp = A_e^{1-m} + w^{-1}A_e^{1-m}w$, and there exist v, $v^\sim \in A_e^{1-m}$ with

$$w^{-1}xw - x^\sim = v^\sim - w^{-1}vw, \qquad w^{-1}(x+v)w = x^\sim + v^\sim. \tag{5.4}$$

Suppose that $e(F[x]/F) = e_0$ and $f(F[x])/F) = f_0$. Since e_0 is the smallest positive integer with $e|e_0m$, $e(F[x^\sim]/F) = e_0$ also. Take e_0th powers in (5.4), noting that $(x+v)^{e_0} \equiv x^{e_0} \bmod A_e^{1-e_0-m}$ (and similarly for $x^\sim$, $v^\sim$). Writing $z = \varpi^{-e_0m}(x+v)^{e_0}$ and $z^\sim = \varpi^{-e_0m}(x^\sim + v^\sim)^{e_0}$, we have

$$w^{-1}zw = z^\sim$$

and $z \equiv \beta \bmod A^1$, $z^\sim \equiv \beta^\sim \bmod A^1$, where β is a root of unity generating the unramified extension F_{f_0} of F and $\beta^\sim \in F_f$ is also a root of unity. Then for every N, $z^{q^{fN}} \equiv z^{q^{f(N-1)}} \bmod A_e^N$, as an easy induction shows. Therefore $z^{q^{fN}} \to \gamma$, where $\gamma^{q^f} = \gamma$ and $\gamma \equiv \beta \bmod A_e^1$. Similarly, $(z^\sim)^{q^{fN}} \to \gamma^\sim$, where $(\gamma^\sim)^{q^f} = \gamma^\sim$, $\gamma^\sim \equiv \beta^\sim \bmod A_e^1$, and $g\gamma g^{-1} = \gamma^\sim$. Let φ, Φ be the minimal polynomials of β, γ respectively. Then $\varphi(\gamma) \equiv \Phi(\gamma) \equiv 0 \bmod A_e^1$. If φ, Φ were distinct, they would be relatively prime over $k = \mathscr{O}_F/P_F$ and we would have $\gamma \equiv 0$, a contradiction. So $\varphi = \Phi$, and β, γ are conjugate. Similarly, $\beta^\sim$, $\gamma^\sim$ are conjugate, so that β, $\beta^\sim$ are conjugate. Therefore x^{e_0}, $x^{\sim e_0}$ are conjugate.

We may thus conjugate (using Lemma 5.2) to arrange that $x^{e_0} = x^{\sim e_0}$. Now it is clear that x, $x^{\sim}$ satisfy the same minimal equation. Therefore they are conjugate in G; Lemma 5.2 implies that they are conjugate by an element of $Z_e K_e$. □

The next result is quite similar to Lemma 2.18 of [15], but the proof given here seems to be shorter and is certainly more in keeping with the methods of this paper.

(5.5) **Lemma.** *Let χ be defined on K_e^j / K_e^{t+1}, with $\chi = \chi_x$ on K_e^t, where $x = \alpha \varpi_e^{-t}$ is such that $F[x]$ is a nicely embedded field. Define $G_{(x)} = \{w \in \mathrm{GL}_n(F) : wx = xw\}$. Let $e(F[x]/F) | e_0$ and $f(F[x]/F) | f_0$. Suppose that b is a power-permutation matrix of some nicely embedded $\mathrm{GL}_{n_0}(E_0)$, with $e(E_0/F) = e_0$ and $f(E_0/F) = f_0$, and that $\chi^b = \chi$ on their common domain. Let k_1, $k \in K_e^{t-j}$. If $\chi^{k_1 b k} = \chi$, then $k_1 b k = k_1' b k'$, with k_1', $k' \in K_e^{t-j+1}(K_e^{t-j} \cap G_{(x)})$.*

Proof. There are some technical issues in the proof that complicate the notation. We need to work with groups K_{n/f_0}^l as well as the groups K_e^g, and we also need to deal with intersections of these groups. We fix j as in the statement of the lemma. Let $u = tf/f_0$ and $h = n/f_0$. We prove the lemma for $K_e^{t-j} \cap K_h^{u-r}$, using induction on r. We write the intersection as $K^{(u-r)}$, and let $'K^{(u-r+1)} = K^{(u-r+1)} K_e^{t-j+1}$; calculations with $K^{(u-r)}$ are usually modulo $'K^{(u-r+1)}$. Let $\sigma = \sigma_h$ and let η_0 be a uniformizer for A_h^1 in $\mathrm{GL}_{n_0}(E_0)$ (where $e_0 f_0 n_0 = n$) such that $\eta_0^{n/e_0 f_0}$ is a prime in E_0 and conjugation by η_0 acts as σ on $m_h^{f_0}(e_0)$. Observe that we can also write $x = \alpha \varpi_h^{-u}$, where $\alpha \in k_{f_0}^{e_0} \subseteq k_{f_0} \subseteq m_h$; this shows that $\chi \equiv 1$ on K_h^{u+1}.

As will soon be clear, in our calculations we are concerned only with k, k_1 modulo $'K^{(u-r+1)}$. (If, e.g., $k' \in {'K^{(u-r+1)}}$, then $k' y k'^{-1} y^{-1} \in \operatorname{Ker} \chi$ for all elements y under consideration.) We therefore write $k = 1 + \delta \eta_0^{u-r}$, $k_1 = 1 + \varepsilon \eta_0^{u-r}$, where $\delta = (\delta_0, \dots, \delta_{h-1})$, $\varepsilon = (\varepsilon_0, \dots, \varepsilon_{h-1})$, and the δ_i, $\varepsilon_i \in M_{f_0}(k)$. Notice that some of the δ_i, ε_i may need to be 0, since in general $K^{(u-r)} \neq K_h^{u-r}$. If, however, δ_i need not be 0, then it can be any element of k_{f_0}, and similarly for ε_i. We write δ_i^* for the element $(\delta_0', \dots, \delta_{h-1}')$ with $\delta_i' = \delta_i$ and all other $\delta_l' = 0$; we define ε_i^* similarly (and use a similar convention for other elements of m_h). Since we are working modulo $'K^{(u-r+1)}$, we may also write $k = (1 + \delta_0^* \eta_0^{u-r}) \cdots (1 + \delta_{h-1}^* \eta_0^{u-r})$, and we may permute the order of the factors. Similar remarks apply to k_1.

We will be looking at the effect of χ on elements $y \in K_e^j \cap K_h^r = L^{(r)}$, for convenience. Write $'L^{(r+1)} = L^{(r+1)} K_e^{j+1}$; it will be clear that we are interested only in y modulo $'L^{(r+1)}$. Thus we set $y = 1 + \gamma \eta_0^r$ and $\gamma = (\gamma_0, \dots, \gamma_{h-1})$, the γ_l arbitrary elements of $M_{f_0}(k)$. Mod $K_h^{u+1} K_e^{j+1}$,

$$k y k^{-1} \equiv y(1 + (\delta \gamma^{\sigma^{u-r}} - \gamma \delta^{\sigma^r}) \eta_0^u).$$

Note that $1 + \delta_i^* \eta_0^{u-r}$ and $1 + \gamma_i^* \eta_0^r$ are (e_0, f_0)-pure.

For $y \in L^{(r)}$, whether $byb^{-1} \in L^{(r)}$ or not is obviously important in determining $\chi^{k_1 b k}(y)$. If $y = 1 + \gamma_i^* \eta_0^r$, then the fact that byb^{-1} is (e_0, f_0)-pure means that exactly one of the following three statements holds (independently of γ_i^*, provided that $\gamma_i^* \neq 0$; we consider only those i for which $\gamma_i^* \neq 0$ for some $y \in L^{(r)}$):

(i) $byb^{-1} \notin L^{(r)}$;
(ii) $byb^{-1} \in {}'L^{(r+1)}$;
(iii) $byb^{-1} \equiv 1 + \beta^*\eta_0^r \bmod {}'L^{(r+1)}$, where $1 + \beta^*\eta_0^r$ is an (e_0, f_0)-pure element (like y) in $L^{(r)}$, and $\beta^* \neq 0$.

We refer to the index i as a "down" index for r if (i) holds, as an "up" index for r if (ii) holds, and as a "steady" index for r if (iii) holds. (We add "with respect to b" when necessary for clarity.)

An example of up, down, and steady indices may help; to keep the matrices of manageable size, the example will not involve a character. Let $n = e_0 = 3$, $f_0 = 1$ (here, $h = e = n$, so that we do not have any complications with pairs of congruence subgroups), $t \ (= u) = 9$, and $j \ (= r) = 7$. Let

$$b \begin{bmatrix} 0 & \varpi & 0 \\ 0 & 0 & 1 \\ \varpi & 0 & 0 \end{bmatrix}, \qquad \varpi = \varpi_F.$$

Since

$$y = 1 + (\gamma_0, \gamma_1, \gamma_2)\varpi_3^7 = 1 + \begin{bmatrix} 0 & \delta_0\varpi^2 & 0 \\ 0 & 0 & \delta_1\varpi^2 \\ \delta_2\varpi^3 & 0 & 0 \end{bmatrix}$$

satisfies $byb^{-1} = 1 + (0, 0, \delta_0)\varpi_e^4 + (0, \delta_2, 0)\varpi_3^7 + (\delta_1, 0, 0)\varpi_e^{10}$, we see that 0 is a down index, 1 an up index, and 2 a steady index for r with respect to b.

If r is a multiple of n_0, then every index is steady for r; in particular, every index is steady for u. Furthermore, since $(\gamma_i^*\eta_0^r)(\delta_{r+i}^*\eta_0^{u-r})$ is of the form $\zeta_i^*\eta_0^u$ (where ζ_i^* is in general nonzero unless γ_i^* or δ_{r+i}^* must be 0), and since $i+r$ is steady for u, we see that:

if i is down for r, then $i+r$ is up for $u-r$;
if i is up for r, then $i+r$ is down for $u-r$;
if i is ready for r, then $i+r$ is steady for $u-r$.

It is also clear that if no element of the form $1+\gamma_i^*\eta_0^r$ $(\gamma_i^* \neq 0)$ is in $L^{(r)}$, then no element of the form $1 + \beta_{i+r}^*\eta_0^{u-r}$ $(\beta_{i+r}^* \neq 0)$ is in $K^{(u-r)}$. It is not hard to see that if i is steady for r with respect to b, so that $b(1+\gamma_i^*\eta_0^r)b^{-1} \equiv 1+\beta_{i'}^*\eta_0^r$ for some i', then i' is steady for r with respect to b^{-1}, and conversely.

The idea of the proof is this: suppose that k has only one nonzero index (previous remarks show that we can reduce to this case). If it is an up index for $u-r$, then $bk = k_0 b$, where $k_0 \in {}'K^{u-r+1}$, and we are done. If it is a down index for $u-r$, then we show directly that $k \in G_{(x)}$. If it is a steady index for $u-r$, then $bk = k_0 b$, where k_0 corresponds to a steady index; we show that the entry for $k_1 k_0$ in that index is in $G_{(x)}{}'K^{u-r+1}$. That takes care of k (and of the steady indices for k_1). To deal with k_1, we need to look at elements of $L^{(r+1)}$ that are conjugated by b into $L^{(r)}$; equivalently and more easily, we repeat the analysis for $(k_1 bk)^{-1}$.

Here are the details. Suppose first that i is down for r. Then $i+r$ is up for $u-r$, and $b(1+\delta_{i+r}^*\eta_0^{u-r})b^{-1} \in {}'K^{(u-r+1)}$. Hence $b(1+\delta_{i+r}^*\eta_0^{u-r}) = k_{(i+r)}b$, where $k_{(i+r)}$ is of the form required by the lemma. Thus we may assume that $\delta_{i+r}^* = 0$ if $i+r$ is up for $u = r$.

If i is up for r, then $i+r$ is down for $u-r$. Write $k = k_{(i+r)}k'k''$, where $k_{(i+r)} = 1+\delta^*_{i+r}\eta_0^{u-r}$ and $k' = 1+(\delta-\delta^*_{i+r})\eta_0^{u-r}$, so that $k'' \in {}'K^{(u-r+1)}$. Let $y = (k'')^{-1}(1+\gamma_i^*\eta_0^{j-1})k'' = (k'')^{-1}y_ik''$, say. Then k' and y commute, so that

$$kyk^{-1} = k_{(i+r)}y_ik^{-1}_{(i+r)} \equiv y_i \mod K_e^u,$$

and $bkyk^{-1}b^{-1} \in K^{(k+1)}$ because i is an up index for r. Since $k_1 \in K^{(u-r)}$, $bkyk^{-1}b^{-1} \equiv k_1bkyk^{-1}b^{-1}k_1^{-1} \mod K_h^{u+1}$. Therefore

$$\chi(bkyk^{-1}b^{-1}) = \chi(k_1bkyk^{-1}b^{-1}k_1^{-1}).$$

By hypothesis, $\chi(bkyk^{-1}b^{-1}) = \chi(kyk^{-1})$. So we must have $\chi(kyk^{-1}) = \chi(y)$, or

$$\chi(k_{(i+r)}y_ik^{-1}_{(i+r)}) = \chi(kyk^{-1}) = \chi(y) = \chi(y_i).$$

The last equality follows from the fact that $k'' \in {}'K^{(u-r+1)}$, so that $y \equiv y_i$ modulo K_h^u. Thus

$$1 = \chi(k_{(i+r)}y_ik^{-1}_{(i+r)}y_i^{-1}) = \chi(1+(\delta^*_{i+r}\gamma_i^{*\sigma^{u-r}} - \gamma_i^*\delta_{i+r}^{*\sigma^u})\eta_0^u) \quad \text{for all } \gamma_i^*,$$

and Lemma 4.7 says that $k_{(i+r)} \in G_{(x)}$ (because $\chi(1+(\delta^*_{i+r}\varepsilon^{\sigma^{u-r}} - \varepsilon\delta_{i+r}^{*\sigma^u})\eta_0^u) = 1$ if $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{h-1})$ and $\varepsilon_i^* = 0$).

Now suppose that i is steady for r and that $b(1+\gamma_i^*\eta_0^r)b^{-1} \equiv 1+\beta_{i'}^*\eta_0^r \mod {}'L^{(r+1)}$. Then $b(1+\delta^*_{r+i}\eta_0^{u-r})b^{-1} \equiv 1+\zeta^*_{r+i'}\eta_0^{u-r}$ (i.e., the only nonzero entries for the coefficient of η_0^{u-r} in the expansion of $b(1+\delta^*_{r+i}\eta_0^{u-r})b^{-1}$ are in entries $\equiv r+i' \mod h/e_0$). As in the case of indices that are up for $u-r$, we may replace $b(1+\delta^*_{r+i}\eta_0^{u-r})$ by $(1+\zeta^*_{r+i'}\eta_0^{u-r})b$ and assume that $\delta_i^* = 0$. We may thus assume that $k = k'k''$, where k' now has entries only in indices that are down for $u-r$ (so that $k' \in G_{(x)} \cap K^{(u-r)}$) and $k'' \in {}'K^{(u-r+1)}$. Then for $y = (k'k'')^{-1}(1+\gamma_i^*\eta_0^r)k'k'' = (k'k'')^{-1}y_ik'k''$, we have $\chi(y) = \chi(y_i)$. So

$$\begin{aligned}\chi(y) &= \chi(k_2bk'k''y(k'k'')^{-1}b^{-1}k_2^{-1}) = \chi(k_2by_ib^{-1}k_2^{-1})\\ &= \chi(by_ib^{-1})\chi(1+(\zeta^*_{i'+r}(\gamma^*_{i'})^{\sigma^{u-r}} - \gamma^*_{i'}(\zeta^*_{i'+r})^{\sigma^r})\eta_0^r),\end{aligned}$$

by a calculation like the one done above. Since $\chi(by_ib^{-1}) = \chi(y_i)$ by hypothesis, we must have

$$\chi(1+(\zeta^*_{i'+r}(\gamma^*_{i'})^{\sigma^{u-r}} - \zeta^*_{i'}(\gamma^*_{i'+r})^{\sigma^r})\eta_0^r) = 1.$$

Because γ_i^* can be arbitrary (with entries only in indices $\equiv i' \mod h/e_0$), we conclude by Lemma 4.7 (as above) that $1+\zeta^*_{i'+r}\eta_0^r \in G_{(x)}$.

We have shown that if $\chi^{k_1bk} = \chi$ on their common domain, then we may assume that $k \in (G_{(x)} \cap K^{(u-r)}){}'K^{(u-r+1)}$ and that $k_1 = k_1'k_1''k_1^*$, where $k_1' \in {}'K^{(u-r+1)}$, $k_1'' \in K^{(r)} \cap G_{(x)}$, and $k_1^* = 1+\varepsilon\eta_0^{u-r}$ with the only nonzero entries for ε in the indices that are up or down for b^{-1}. Apply the above argument to $k^{-1}b^{-1}k_1^{-1}$. Evidently $k^{-1}\cdot(k_1'k_1'')^{-1} \in (G_{(x)} \cap K^{(u-r)}){}'K^{(u-r+1)}$, so that we need only concern ourselves with $b^{-1}(k_1^*)^{-1}$. Since

$$(k_1^*)^{-1} \equiv 1-\varepsilon\eta_0^{u-r} \mod (G_{(x)} \cap K^{(u-r)}){}'K^{(u-r+1)},$$

the above proof shows that $b^{-1}(k_1^*)^{-1} = k_0^{-1}b^{-1}k_0'^{-1}$, where k_0^{-1}, $k_0'^{-1} \in {}'K^{(u-r+1)}$. Hence $k_1^*b = k_0'bk_0$, where k_0, $k_0' \in {}'K^{(u-r+1)}$; the lemma follows. □

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We now start the construction of the representations ρ_1 that induce to supercuspidals. We begin with the sequence of triples (s_i, e_i, f_i), $1 \leq i \leq r$, satisfying the properties of (1.3). For the moment, we assume that $s_1 > 0$; the case $s_1 = 0$ will be dealt with later. Let $t_i = s_i/f$ and $n_i = n/e_i f_i$; set $e_0 = f_0 = 1$. For $t_i > 0$, set $t_1' = [t_i/2] + 1$. Let r_0 be the largest index with $t_{r_0} \geq 2t_{r_0+1}$ or, equivalently, $t'_{r_0} > t_{r_0+1}$ (if no such index exists, set $r_0 = r$). Write $t_i'' = t_i + 1 - t_i'$, so that $t_i'' = t_i'$ if t_i is odd and $t_i'' = t_i' - 1$ if t_i is even. If $t_i = 0$ (so that $i = r$), set $t_i'' = t_i' = 0$. Define numbers C_j by

$$C_{t_i} = \frac{f_{i-1}}{f_i} \sum_{df_{f-i}|f_i} \mu(f_i/f_{i-1}d)(q^{f_{i-1}d} - 1), \qquad \mu = \text{Möbius function};$$

$$C_j = \begin{cases} 1 & \text{if } t_{i+1} < j < t_i \text{ and } e \nmid je_i; \\ q^{f_i} & \text{if } t_{i+1} < j < t_i \text{ and } e | je_i; \\ q^f & \text{if } 0 < j < t_r; \\ q^f - 1 & \text{if } j = 0 < t_r. \end{cases}$$

(The C_i depend on the (s_i, e_i, f_i), but we do not indicate the dependence in the notation.)

The major result in the construction, given next, is analogous to Theorem 3.1 of [4]. Before stating it, we give an indication of the strategy to be followed in the construction. We begin with a character χ defined on $K_e^{t_1}$ and trivial on $K_e^{t_1+1}$. Such a χ can be defined by

$$\chi(1+y) = \psi \circ \operatorname{Tr}(\alpha \varpi_e^{-t_1} y) \quad \forall y \in A_e^{t_1},$$

for a unique $\alpha \in m_e$. (In fact, $\chi(1+y) = \psi \circ \operatorname{Tr}(xy)$ for any $x \equiv \alpha\varpi_e^{-t_1}$ modulo $A_e^{1-s_1}$. We are making the simplest choice of x, in some sense.) We require that $\alpha\varpi_e^{-t_1}$ generate a field $E_{(t_1)}$ with $e(E_{(t_1)}/F) = e_1$ and $f(E_{(t_1)}/F) = f_1$. Up to conjugacy, this turns out to give C_{t_1} choices for α. We then compute the elements $g \in G$ such that $\chi(gyg^{-1}) = \chi(y)$ on their common domain (we say that such an element commutes with χ). This set includes $G_{(t_1)}$, the group of elements commuting with $E_{(t_1)}$. ($G_{(t_1)} \cong \mathrm{GL}_{n/e_1 f_1}(E_{(t_1)})$.)

The next step is to show that there is an extension χ_0 of χ to $K_e^{t_1-1}$ such that g commutes with χ_0 if $g \in G_{(t_1)}$. The point of this step is that any extension of χ to $K_e^{t_1-1}$ must be of the form $\chi = \chi_0\chi_1$, where χ_1 is trivial on $K_e^{t_1}$. Therefore we can find a convenient expression for χ_1 like the one given above for χ on $K_e^{t_1}$. It turns out that two extensions $\chi_0\chi_1$ and $\chi_0\chi_1^{\sim}$ are conjugate if they agree on $K_e^{t_1-1} \cap G_{(t_1)}$. Therefore we need only look at χ_1 on this subgroup. It is given there by

$$\chi(1+y) = \psi \circ \operatorname{Tr}(\alpha_1 \eta_{(t_1)}^{-t_1} y) \quad \forall y \in A_e^{t_1-1} \cap M_{(t_1)},$$

where $M_{(t_1)} \cong M_{n/e_1 f_1}(E_{(t_1)})$ is the set of elements in $M_n(F)$ commuting with $E_{(t_1)}$, $\alpha_1 \in m_e \cap M_{(t_1)}$, and $\eta_{(t_1)} \in G_{(t_1)}$ is an element that plays a role in $G_{(t_1)}$ like that played by ϖ_e in G. If $t_2 < t_1 - 1$, we require that $\alpha_1\eta_{(t_1)}^{-t_1} \in E_{(t_1)}$; if $t_2 = t_1 - 1$, we require that $\alpha_1\eta_{(t_1)}^{-t_1}$ generate an extension field of $E_{(t_1)}$ whose ramification index and residue class degree (over F) are e_2 and f_2 respectively.

It turns out that there are C_{t_1-1} choices of α_1 (up to conjugacy) in each case. Let $\chi = \chi_0\chi_1$. We now compute the set of elements of G that commute with χ. In the first case $(t_2 < t_1 - 1)$, this set contains the group $G_{(t_1-1)}$ of all elements of G commuting with some field $E_{(t_1-1)}$ such that $e(E_{(t_1-1)}/F) = e_1$ and $f(E_{(t_1-1)}/F) = f_1$, but we need not have $E_{(t_1-1)} = E_{(t_1)}$. Similarly, in the case where $t_2 = t_1 - 1$ there is a field $E_{(t_1-1)} = E_{(t_2)}$ such that $e(E_{(t_2)}/F) = e_2$, $f(E_{(t_2)}/F) = f_2$, and any element g commuting with $E_{(t_2)}$ commutes with χ, but $E_{(t_2)}$ need not be the extension field of E_1 generated by $\alpha_1\eta_{(t_1)}^{-t_1}$. It is the fact that these fields can vary in odd ways that makes the wild case $(p|n)$ more difficult than the tame case $(p\nmid n)$; we say more about the tame case later.

Having found $E_{(t_1-1)}$, we now continue to $K_e^{t_1-2}$ and repeat the procedure. There are some variations that occur as we continue the induction. For example, $K_e^{t_1'}/K_e^{t_1+1}$ is an Abelian group, but $K_e^{t_1'-1}/K_e^{t_1+1}$ is not; we cannot extend χ as a character to all of $K_e^{t_1'-1}$. This means that as we go along we need to extend the definition of the group H on which χ is defined. As this sketch undoubtedly suggests, there are large numbers of other details to verify in the course of the proof.

The reader may wonder why this procedure is natural or even reasonable. It was prompted by the need to solve three technical problems. First of all, supercuspidal representations should correspond somehow with characters associated to maximal anisotropic tori, but in the wild case nonconjugate tori (= maximal embedded subfields) can lie very close together, and the characters naturally associated with one of these tori tend to be characters on subgroups of GL_n that may be trivial on the intersection with the associated torus itself. The above procedure does associate χ with tori, by associating χ with the fields $E_{(j)}$; one can think of the induction as providing a sequence of fields that approximate the field associated with χ. A second problem is that describing the character χ becomes increasingly difficult as one goes on. On $K_e^{t_1'}$, $\chi(1+y)$ can be given as $\psi\circ\mathrm{Tr}(xy)$ for an appropriate x, but for larger groups there seems no easy way to describe χ. The procedure given here obviates the need for a detailed description of χ; one simply needs to describe how χ extends one layer at a time, and the necessary information is given by χ_0. Thirdly, describing the set of elements commuting with χ becomes increasingly difficult as time goes on. It becomes manageable in this inductive procedure because if we have defined χ on $H\cap K_e^{j-1}$, then any element that commutes with χ there also commutes with $\chi|_{H\cap K_e^j}$. This simplifies computations considerably. In fact, we will see that virtually every calculation reduces to one on A_e/A_e^1.

(6.1) **Theorem.** *There exist* $\prod_{j=t'_{r_0}}^{t_1} C_j$ *choices of fields* $E_1, \dots, E_{r_0}$, *elements* $\eta_{(t_1)}, \dots, \eta_{(t'_{r_0})} \in Z_eK_e$, *and characters* χ *on subgroups* H_0, *nonconjugate under* Z_eK_e *and trivial on* $K_e^{t_1+1}$, *with the following properties* (*in the rest of this statement we use the notational convention that for an index* j, i *is the largest index with* $t_i \geq j$):

(1) $\eta_{(j)}\gamma\eta_{(j)}^{-1} = \gamma^\sigma$ *for all* $\gamma \in m_e^{f_i}(e_i)$.

(2) $\eta_{(j)}$ *generates* A_e^1 (*as an ideal of* A_e). *Indeed,* $\eta_{(j)} \equiv \varepsilon_j\varpi_e \bmod A_e^2$, *with* $\varepsilon_j \in (k_{f_i}^\times)^e$.

(3) $E_{(j)} = F_{f_i}[\eta_{(j)}^{e/e_i}]$ *is a nicely embedded field extension of* F *with ramification index* e_i *and residue class degree* f_i. (*Recall*: $[F_{f_i} : F] = f_i$, *and* F_{f_i} *is unramified over* F.) *This means also that* $\eta_{(j)}$ *and* $\eta_{(j)}^{e/e_i}$ *are related as described at the start of* §2.

(4) *Let* $M_{(j)} = M_{n_i}(E_{(j)}) \subseteq G$ *and* $M_i = M_{(t_{i+1}+1)}$ (*and* $M_{r_0} = M_{(t'_{r_0})}$). *Then* $M_{(j)}$ *is generated over* $E_{(j)}$ *by* $\eta_{(j)}$ *and* $m_e^{f_i}(e_i)$.

Write $G_{(j)} =$ (*group of invertible elements of* $M_{(j)}$) $= \mathrm{GL}_{n_i}(E_{(j)})$, $G_i =$ *group of invertible elements of* $M_i = M_{(t_{i+1}+1)}$, $E_i = E_{(t_{i+1}+1)}$ (*for* $i < r_0$), $G_{r_0} = G_{(t'_{r_0})}$, *etc.*

(5) *If* $j \neq t_i$, *then* $\eta_{(j)}$ *is of the form* $\eta_{(j+1)} + y_{i-1} + \cdots + y_0$, *with* $y_g \in M_g \cap A_e^{t_{g+1}-j+1}$. *If* $j = t_i$, *then* $\eta_{(j)}$ *is of the form* $\zeta_i \eta_{i-1} + y_{i-2} + \cdots + y_0$, *where the* y_g *are as before and* $\zeta_i \in (k_{f_i}^{\times})^e$.

(6) $H_0 = K_e^{t'_1}(K_e^{t'_2} \cap G_1) \cdots (K_e^{t'_{r_0}} \cap G_{r_0-1})$, *and*

$$H_0 \cap K_e^j = K_e^{t'_1}(K_e^{t'_2} \cap G_1) \cdots (K_e^{t'_h} \cap G_{h-1})(K_e^j \cap G_h),$$

where h *is the index with* $t'_{h+1} \leq j < t'_h$. (*We set* $t_0 = t'_0 = +\infty$.) *In particular,* $H_0 \cap K_e^j / H_0 \cap K_e^{j+1} \cong K_e^j \cap G_h / K_e^{j+1} \cap G_h$.

We define $H_0^j = H_0 \cap K_e^j$.

(7) *The set of elements in* $Z_e K_e$ *commuting with* $\chi|_{H_0^j}$ *is*

$$K_e^{c_1}(K_e^{c_2} \cap G_1) \cdots (K_e^{c_i} \cap G_{i-1})(Z_e K_e \cap G_{(j)}),$$

where c_l *is the smaller of* t''_l *and* $t_l + 1 - j$; *this set is a group and normalizes* H_0^j. *Furthermore,* $\chi(gyg^{-1}) = \chi(y)$ *whenever* $g \in \mathrm{GL}_{n_i}(E_{(j)}) = G_{(j)}$ *and* y, $gyg^{-1} \in H_0^j$ (*i.e.,* $G_{(j)}$ *commutes with* $\chi|_{H_0^j}$).

(8) *Let* $t'_{h+1} \leq t_i < t'_h$, *so that* G_{i-1} *commutes with* χ *on* $H_0^{t_i+1}$ *and* $H_0^{t_i}/H_0^{t_i+1} \cong G_h \cap K^{t_i}/G_h \cap K^{t_i+1}$. *Then one can write* $\chi|_{H_0^{t_i}}$ *as* $\chi_{0,i}\chi_{1,i}$, *where*

(i) G_{i-1} *commutes with* $\chi_{0,i}$;

(ii) $\chi_{1,i}$ *is trivial on* $H_0^{t_i+1}$;

(iii) *On* $H_0^{t_i} \cap G_{i-1}$ (*see the note below*), *we have*

$$\chi_{1,i}(1 + \gamma\eta_{i-1}^{t_i}) = \psi \circ \mathrm{Tr}^{(e_{i-1})}(\alpha_i^{\sigma^{t_i}}\gamma), \qquad \gamma \in m_e^{f_{i-1}}(e_{i-1}),$$

where $E_{i-1}[\alpha_i \eta_{i-1}^{-t_i}] = E_{\widetilde{(t_i)}}$ *is a nicely embedded field satisfying* $e(E_{\widetilde{(t_i)}}/F) = e_i$, $f(E_{\widetilde{(t_i)}}/F) = f_i$;

(iv) *the matrix algebra* $M_{\widetilde{(t_i)}}$ *of elements commuting with* $E_{\widetilde{(t_i)}}$ *has an element* $\eta_{\widetilde{(t_i)}}$ *generating* A_e^1 *such that* $\eta_{\widetilde{(t_i)}}$, $m_e^{f_i}(e_i)$ *generate* $M_{\widetilde{(t_i)}}$ *as in Proposition* 2.1, *and* $\eta_{\widetilde{(t_i)}} \equiv \eta_{(t_i)} \bmod A_e^2$.

Note that in (ii) *and* (iii) *we are describing* $\chi_{1,i}$ *only on part of its domain, namely on* $H_0^{t_i+1}(H_0^{t_i} \cap G_{i-1}) \subseteq H_0^{t_i} = H^{t_i+1}(H_0^{t_i} \cap G_{h-1})$.

(9) *Suppose that* $j_0 < j$; *let* $g \geq i$ *satisfy* $e|j_0 e_g$. *Let* $\chi^{\#}$ *be a character on* $G_{i-1} \cap K_0^{j_0}$ *trivial on* $G_{i-1} \cap K_e^{j_0+1}$ *and on* $G_i \cap K_e^{j_0}$, *of the form*

$$\chi^{\#}(1 + \gamma\eta_{i-1}^{j_0}) = \psi \circ \mathrm{Tr}(\beta\gamma^{\sigma^{-j_0}}), \qquad \beta \in k_{f_g}^e.$$

Then there exists $w = 1 + \delta_i \eta_{i-1}^{t_i - j_0}$, *with* $\delta_i \in k_{f_g}^e$ *and* $\delta_i^{\sigma^{e/e_{i-1}}} = \delta_i$, *such that* $\chi(wyw^{-1}y^{-1}) = \chi^{\#}(y)$, $y \in G_{i-1} \cap K_e^{j_0}$.

(10) *If* $t_i' < j \leq t_i$, *then any extension of* $\chi|_{H_0^{t_i}}$ *to a character on* H_0^j *extends to a character on* $H_0^{t_i'}$.

(11) *The* χ *are nonconjugate under* $Z_e K_e$, *in that if* χ_1, χ_2 *are distinct characters on* $H_{0,1}$, $H_{0,2}$ *respectively, then for any* $z \in Z_e K_e$, χ_1^x *and* χ_2 *do not agree on their common domain.*

Proof. The proof is by backwards induction on j; it is long and complicated. There are five main steps:

1. Proving the theory for $j = t_1$.
2. Proving a technical lemma that says roughly that if we have χ defined on H_0^j so that $G_{(j)}$ commutes with χ, then χ has an extension χ_0 to H_0^{j-1} such that $G_{(j)}$ commutes with χ_0. The value of this lemma, as noted in the remarks before the statement of the theorem, is that any extension of χ to H_0^{j-1} differs from χ_0 by a character that is trivial on H_0^j. This gives us a concrete way of describing all such extensions.
3. Proving the inductive step (to $j-1$) in the case where $e \nmid (j-1)e_i$ (as before, i is the largest index with $j \leq t_i$). This is the easiest step.
4. Proving the inductive step in the case where $e | (j-1)e_i$, but $j - 1 \neq t_{i+1}$.
5. Proving the theorem in the case where $j - 1 = t_{i+1}$.

In this section we do the first two steps; the rest of the proof is given in §7.

Before beginning the work of the proof, we shall set some notation and attempt to explain the meaning of (1)–(11). Properties (1)–(4) give a useful working description of the groups $G_{(j)}$ and the fields $E_{(j)}$, and (5) shows that these groups are related in a way that makes the results of §3 applicable. Property (6) gives a description of the group H_0 on which the character is defined. (Note that H_0 is not given at the start of the theorem, but is instead defined inductively in the course of the proof. When we reach level t_i, $H_0^{t_i'}$ is defined; the definition involves $G_{i-1} = G_{(t_i+1)}$ in a critical way. Property (5) and Lemma 3.1 show that, for instance, the definition of $H_0^{t_{i-1}'}$ does not change when we later define $H_0^{t_1'}$.) Property (7) gives important information about the elements of G that commute with $\chi|_{H_0^j}$; a full description of this set is given later, in Theorem 8.1. Properties (8) and (9) give technical information about χ used in proving some of the other properties; specifically, (8) is used so that we can apply Lemmas 4.7 and 4.8 when considering elements commuting with χ (as we must when proving (7), and (9) is used to construct $\eta_{(j-1)}$ from $\eta_{(j)}$. Property (10) means that we have a certain amount of freedom in extending χ. The nonconjugacy statement in (11) will be used to show that the supercuspidals constructed are all distinct.

Our list of properties is redundant, in that some of the properties imply others. For instance, (8) $\Rightarrow$ (9) by Lemma 4.7, and (5) $\Rightarrow$ (6) by Lemma 3.1 and Corollary 3.2 (the first part of (6) is definition). Next, (5) also implies that the set in (7) is a group, because of Lemma 3.1, since the c_i in (7) always satisfy $c_{i-1} - c_i \leq t_{i-1} - t_i$. Finally, (6) and (7) imply (10), since the commutator subgroup $(H_0^{t_i'}, H_0^{t_i'}) \subseteq H_0^{t_i}$ and an argument using (4.3) shows that $\chi \equiv 1$ on the commutator. (If $x = 1 + \gamma\eta_i^r$ and $y = 1 + \delta\eta_i^s$, where r, $s \geq t_i'$ and γ, $\delta \in (m_e^{f_i}(e_i))^\times$, then all words in $(x-1)$ and $(y-1)$ are in $H_0^{t_i}$, and Lemma 4.2 applies.) Thus we shall not prove (6), (9), or (10) in the induction.

In the inductive part of the proof, we assume the result for j; i is the largest index with $t_i \geq j$, and h is the index with $t_{h+1}' < j \leq t_h'$. The reason for listing h is that $H_0^{j-1}/H_0^j \cong G_h \cap K_e^{j-1}/G_h \cap K_e^j \cong m_e^{f_h}(e_h)$; however, h does not play a major role in any arguments. We generally use g and l as indices.

At times, we shall use (4.4) to deal with the case where k has two elements. What we need to know is that for some large prime N, if we work with $\mathrm{GL}_n(F_N)$ (where F_N is the unramified extension of degree N), then we can perform the construction (on the composita of the fields $E_{(j)}$ with F_N), getting a character $\chi^\sim$ on a group $H_0^\sim$ such that $H_0 = H_0^\sim \cap \mathrm{GL}_n(F)$ and $\chi^\sim|_{H_0} = \chi$. It should not be hard to see that this is always the case.

When $j-1$ is not one of the "jump indices" t_{i+1}, the objects $E_{(j-1)}$, $G_{(j-1)}$, etc. satisfy the properties for $E_{(j)}$, $G_{(j)}$, etc. It may be used to regard, e.g., $E_{(t_1)}$, $E_{(t_1-1)}, \ldots, E_{(t_2+2)}$ as successive approximations to $E_1 = E_{(t_2+1)}$.

If $p \nmid n$ (the "tamely ramified" case, treated in [10]), it turns out that we can always take $E_{(j-1)} = E_{(j)}$ when $j-1$ is not a jump index. (This is the point of the "geometrical" lemmas in the first part of [10], which show, e.g., that $M_{(t_1)} \oplus M_{(t_1)}^\perp = M$, where $M_{(t_1)}^\perp = \{x \in M\colon \mathrm{Tr}(xy) = 0 \;\; \forall y \in M_{(t_1)}\}$.) Furthermore, at a jump index, we have $E_{(j-1)} = E_{(j-1)}^\sim$ (see property (8), (iii)), so that $E_1 \subset E_2 \subset \cdots \subset E_{r_0}$. This simplifies the description of χ and also simplifies many details of the construction. The reader may wish to compare the construction that follows with that of [10] when both apply.

The proof is so arranged that at most points in the argument we need be concerned only with a character χ_1 (related to χ) defined on some H_0^h and trivial on H_0^{h+1}. (This may indicate the importance of (8).) We often need an argument, using Lemma 4.2, to reduce to this situation, but it may help to keep this organizational principle in mind.

We now give the proof for the case $j = t_1$. The characters on $K_e^{t_1}$ trivial on $K_e^{t_1+1}$ are of the form

$$\chi(1+y) = \psi \circ \mathrm{Tr}(\alpha_1 \varpi_e^{-t_1} y), \qquad y \in A_e^{t_1} \text{ and } \alpha_1 \in m_e.$$

We require that $\alpha_1 \in k_{f_1}^e$, that $F[\alpha_1\varpi_e^{-t_1}] = E_{(t_1)}$ have ramification index e_1 and residue class degree f_1, and that $F_{f_1} \subseteq E_{(t_1)}$. Up to conjugacy, the number of choices for α_1 is the number of primitive elements for k_{f_1}/k, with elements equivalent if they have the same minimal equation, namely C_{t_1}. We fix one representative for each conjugacy class. Notice that $(\alpha_1\varpi_e^{-t_1})^{e_1} = \gamma_1\varpi_F^{-a}$, say, where the entries of γ_1 are all equal. Since the entries of α_1 all commute and the lth entry of γ_1 depends only on those entries of α_1 with indices $\equiv l \bmod e/e_1$, so that $\gamma_1 = N_{e_1}(\alpha_1)$, we may arrange to have the first e/e_1 entries of α_1 equal, the next e/e_1 entries equal, and so on. Pick one α_1 of this form in each conjugacy class; for this choice, $\alpha_1\varpi_e^{-t_1}$ commutes with every (e_1, f_1)-permutation matrix, and $E_{(t_1)}$ is nicely embedded. For appropriate a, $b \in \mathbb{Z}$, $(\alpha_1\varpi_e^{-t_1})^a\varpi_F^b$ is of the form $\delta\varpi_e^{e/e_1}$, since $(t_1, e) = e/e_1$; $F[\delta\varpi_e^{e/e_1}] = E_{(t_1)}$. Furthermore, $\delta = (\delta_0, \delta_1, \ldots, \delta_{e-1}) \in (k_f^\times)^e$, with $\delta_0 = \delta_1 = \cdots = \delta_{e/e_1-1}$, $\delta_{e/e_1} = \cdots = \delta_{2e/e_1-1}$, etc. Let $\gamma_1 = (1, \ldots, 1, \delta_0, 1, \ldots, \delta_{e/e_1}, \ldots)$, where there are $e/e_1 - 1$ ones between the δs, and set $\eta_{(t_1)} = \gamma_1 \varpi_e$. Then $\eta_{(t_1)}^{e/e_1} = \delta \varpi_e^{e/e_1}$, so that $F[\eta_{(t_1)}^{e/e_1}] = E_{(t_1)}$. We can easily insure that $\gamma_1 \in k_{f_1}^e$.

We next check properties (1)–(11). Property (1) is easy, since α_1 commutes with $m_e^{f_1}$; (2) is immediate, and it is also clear that (3) holds. As for (4), note that $\eta_{(t_1)}$ and $m^{f_1}(e_1)$ commute with $\alpha_1 \varpi_e^{-t_1}$ and are therefore in $M_{(t_1)}$. They generate a vector space of dimension n_1^2 over $E_{(t_1)}$, and hence generate $M_{(j)}$. (5) is vacuous, and we checked (11) above; for (8), let $\chi_{0,1} \equiv 1$. For the second part of (7), set $\alpha_1 \varpi_e^{-t_1} = u$. If $g \in G_{(t_1)}$, then $[g, u] = 0$. So if y, $gyg^{-1} \in A_e^{t_1}$, then $\chi(1 + gyg^{-1}) = \psi \circ \mathrm{Tr}(ugyg^{-1}) = \psi \circ \mathrm{Tr}(g^{-1}ugy) = \psi \circ \mathrm{Tr}(uy) = \chi(1+y)$. The first part of (7) is a bit more work. It is easy to check that K_e^1 and $Z_e K_e \cap G_{(t_1)}$ commute with $\chi|_{K_e^{t_1}}$; note that $(K_e^1, K_e^{t_1}) \subseteq K_e^{t_1+1}$. Conversely, any element of $K_e Z_e$ can be written uniquely as

$$w = (1 + \beta_1 \varpi + \cdots)\beta_0 \varpi^{j_0} = w_1 w_0, \qquad \beta_i \in m_e \text{ and } \beta_0 \in m_e^\times .$$

We know that w_1 commutes with χ; thus the hypothesis says that $\chi^{w_0} = \chi$. By Lemma 4.8, w_0 commutes with u.

We now give the technical result of the second step. It is analogous to Lemma 3.8 of [4].

(6.2) **Lemma.** *Retain the notation of Theorem* 6.1. *Let* χ (*as above*) *be defined on* H_0^j, $j > t'_{r_0}$, *and let* i, h *be as defined above in the proof of Theorem* 6.1. *Assume that* $G_{(j)}$ *commutes with* χ. *Then* χ *has an extension* χ_0 *to* H_0^{j-1} *such that* $G_{(j)}$ *commutes with* χ_0.

Remark. We shall actually prove slightly more about χ_0, since we shall show that $G_{(j)}$ commutes with χ_0 once a fairly small subset of $G_{(j)}$ commutes with χ_0. This may be useful elsewhere. In addition, the proof uses little about $G_{(j)}$ except that $E_{(j)}$ is nicely embedded, $G_{(j)}$ commutes with χ, and elements of $G_{(j)}$ have the "normalizing" property described at the start of the proof; thus we can often apply the result to groups other than $G_{(j)}$.

Proof of the lemma. In this proof, we assume that $q \neq 2$; when $q = 2$, we need to modify the proof as noted earlier.

We begin by indicating something about when elements of $G_{(j)}$ "normalize" H_0^{j-1}. Specifically, we show that if $x \in G_{(j)}$, $y \in H_0^{j-1}$, and $xyx^{-1} \in K_e^{j-1}$, then $xyx^{-1} \in H_0^{j-1}$. From (5) and (6) (or the proof of (7)), $G_{(j)} \cap K_e$ normalizes H_0^{j-1}. Thus we may assume that $x = b$ is a power-permutation matrix in $G_{(j)}$. We may also assume that $y \in G_l \cap K_e^{t'_l+1}$ for some l. Suppose that $l = h - 1$. Repeated use of (5) shows that $\eta_{(j)}$ can be written in the form

$$\eta_{(j)} = \zeta_h \eta_h + y_h + y_{h-1} + \cdots + y_0, \qquad \zeta_{h-1} \in (k_{f_i}^\times)^e,$$

where $y_g \in M_g \cap A_e^{t_{g+1}-t_{h+1}+2}$. [From (5), we get $\eta_{(j)} = \zeta_{(j+1)}\eta_{(j+1)} + y'_{i-1} + \cdots + y'_0$, $y'_g \in M_g \cap A_e^{t_{g+1}-j+1}$. Apply (5) to the first term; we get $\eta_{(j)} = \zeta_{(j+2)}\eta_{(j+2)} + y'_{i-1} + \cdots + y'_0$, $y'_g \in M_g \cap A_e^{t_{g+1}-j}$, where the y'_g may have changed from one line to the next. Repeating this procedure, we eventually get

$$\eta_{(j)} = \zeta_{i-1}\eta_{i-1} + y'_{i-1} + \cdots + y'_0, \qquad y'_g \in M_g \cap A_e^{t_{g+1}-t_i+2};$$

again, the y'_g may have changed. Now use (5) and Lemma 3.3 to replace η_{i-1} with $\zeta_{(t_i+2)}\eta_{(t_i+2)} + y''_{i-2} + \cdots + y''_0$, $y'_{i-1} = z_{(t_i+2)} + z_{i-1} + \cdots + z_0$, where $y''_g \in M_g \cap A_e^{t_{g+1}-t_i+1}$, $z_{(t_i+2)} \in M_{(t_i+2)} \cap A_e^2$, and $z_g \in M_g \cap A_e^{t_{g+1}-t_i+2}$. Combining terms, we have

$$\eta_{(j)} = \zeta_{(t_i+2)}\eta_{(t_i+2)} + y'_{(t_i+2)} + y'_{i-2} + \cdots + y'_0,$$

where $\zeta_{(t_i+2)} \in (k_{f_i}^\times)^e$, $y'_{(t_i+2)} \in M_{(t_i+2)} \cap A_e^2$, and $y'_g \in M_g \cap A_e^{t_{g+1}-t_i+1}$. Continue inductively; we get

$$\eta_{(j)} = \zeta_h\eta_h + y'_h + \cdots + y'_0, \qquad y'_g \in M_g \cap A_e^{t_{g+1}-t_{h+1}+2},$$

as desired.]

Now use Lemma 3.3: there is a power-permutation matrix b' (for the compositum of F_{f_i} and E_h) such that for some

$$k' \in K_e^1 \cap G_h(K_e^{t_{h-1}-t_{h+1}+1} \cap G_{h-2}) \cdots K_e^{t_1-t_{h+1}+1}, \quad b = k'b'.$$

It is obvious that $b'yb'^{-1} \in H_0^{j-1}$ if it is in K_e^{j-1}, and Corollary 3.2 says that k normalizes H_0^{j-1}. The argument for the other $G_l \cap K_e^{t'_{l+1}}$ is essentially the same.

It may be worth noting why the lemma is true when $h = 0$ (though what follows is not a proof). Then $j > t'_1$ and $H_0^{j-1} = K_e^{j-1}$, and we can write χ on K_e^j as

$$(6.3) \qquad \chi(1+y) = \psi \circ \mathrm{Tr}(xy), \qquad x \in A_e^{-t_1}, \ x \text{ determined } \bmod A_e^{1-j}.$$

It is possible (though not obvious in our inductive process) to choose x so that $F[x] = E_{(j)}$. Assuming this, the proof of the lemma is easy: use (6.3) (with this same x) to define χ on K_e^{j-1}. In fact, we could also replace x by $x + x_0$, where $x_0 \in A_e^{1-j} \cap E_{(j)}$. This choice will change χ on K_e^{j-1} only if $x_0 \notin A_e^{2-j} \cap E_{(j)}$, which suggests that the extension is unique if $A_e^{1-j} \cap E_{(j)} = A_e^{2-j} \cap E_{(j)}$. This fact may help explain the division below into cases.

Before entering into the body of the proof, we simplify notation. We will generally be concerned with elements representing $H_0^{j-1}/H_0^j \cong m_e^{f_h}(e_h)$. Since all the elements we will be concerned with will commute with k_{f_h}, we may as well assume that $f_h = 1$. Similarly, the elements we deal with will all have the e/e_h-periodicity of elements of $m_e^{f_h}(e_h)$, and it will therefore not affect the proof if we assume that $e_h = 1$. The real effect of these assumptions is that we can work with m_e instead of $m_e^{f_h}(e_h)$. In a sense, we are assuming that $h = 0$, and the discussion of power-permutation matrices given at the start of the proof shows that this assumption does not affect whether the elements we deal with in the proof actually lie in H_0^{j-1}. Of course, we must be careful to give a proof that is valid without these simplifications; for instance, the reasoning given above would be impermissible.

Because we want to use the decomposition $G_{(j)} = B_{(j)}W_{(j)}B_{(j)}$, where $B_{(j)}$ is the Iwahori subgroup and $W_{(j)}$ is the group of power-permutation matrices, we need to deal with another set of congruence subgroups. Set $d = n/f_i$, and write $H_0^{j-1} \cap K_{n/f_i}^l = H_l$. We define χ_0 to H_l by backwards induction, assuming that it is already defined on H_{l-1} (to conserve notation, the extension to H_{l-1} will be called χ). Write d for n/f_i, and let $\eta \in G_{(j)}$ generate A_d^1 and induce σ on $m_d^{f_i}(e_i)$. The coset representatives for H_l/H_{l+1} can be taken as elements $1+\gamma\eta^l$, $\gamma = (\gamma_0, \dots, \gamma_{d-1}) \in m_d$; when $d \neq e$, some of the γ_j may have to be 0, but for a given index h either γ_h must be 0 or there is no restriction on γ_j (except, of course, that it commute with k_{f_i}; since we are in $m_d \cong M_{f_i}(k)^{n/f_i}$, this means that $\gamma_j \in k_{f_i}$). In particular, $(k_{f_i}^\times)^d$ normalizes H_l. However, if $f/f_i|l$, then there is no requirement that any γ_j be 0.

The proof divides into several large steps:

1. We find an extension χ_1 to H_l such that $(k_{f_i}^\times)^d \cap m_d^1(e_i)$ $(\cong K_d \cap G_{(j)}/K_d^1 \cap G_{(j)})$ commutes with χ_1.

2. If l is not divisible by d/e_i, then χ_1 is the only extension of χ with the property in 1. We use this to show that every power-permutation matrix in $G_{(j)}$ commutes with χ. In particular, η commutes with χ, and Lemma 4.2 now implies that $K_d^1 \cap G_{(j)} = B_{(j)}$ commutes with χ. ($B_{(j)} \subseteq K_e \cap G_{(j)}$ normalizes H_l.) That proves the lemma in this case, with $\chi = \chi_0$.

3a. If l is divisible by d/e_i, then χ_1 is not unique, but we can choose coset representatives for H_l/H_{l+1} that remain in H_l under conjugation by any element of $W_{(j)}$. We prove next that there is an extension χ_0 such that η commutes with χ_0; χ_0 is also not unique, but it is easy to describe the other possible choices. We also show that any "diagonal" matrix whose diagonal elements are powers of η commutes with χ_0.

3b. In view of 3a, we need only prove that permutation matrices commute with χ_0. (This uses the fact that we have coset representatives as described in 3a; we therefore need to prove that $\chi_0^b = \chi_0$ only for a set of generators b for $W_{(j)}$.) We prove this last fact to complete the proof.

1. Write $(k_{f_i}^\times)^{d/e_i}$ for $(k_{f_i}^\times)^d \cap m_d^1(e_i)$ $(\cong (k_{f_i}^\times)^{d/e_i})$. The extensions of χ to H_l form an affine space of cardinality $[H_l : H_{l+1}]$, a power of p, and $(k_{f_i}^\times)^{d/e_i}$ has order prime to p. Let $U_1, \dots, U_m$ be the orbits of the extensions under conjugation by $(k_{f_i}^\times)^{d/e_i}$, and let a_j = cardinality of U_j. Since $\sum_{j=1}^m a_j$ is a power of q and each a_j is prime to p, there are integers b_j with $\sum_{j=1}^m a_j b_j = 1$. Then set

$$\chi_1 = \prod_{j=1}^m \prod_{\chi' \in U_j} \chi'^{b_j};$$

χ_1 clearly has the desired property, and (1) is done.

2. We assume that $d/e_i \nmid l$. Then χ_1 is the only extension of χ commuting with χ. For every extension of χ to H_l is of the form $\chi^\sim(y) = \chi_1(y)\chi_2(y)$, where χ_1 is as above and χ_2 is trivial on H_{l+1}. Thus $\chi_2(1+\gamma\eta^l) = \psi \circ \mathrm{Tr}(\alpha^{\sigma^l}\gamma)$ for some $\alpha \in m_d$, and the same calculation as for Lemma 4.8 shows that $(k_{f_i}^\times)^{d/e_i}$ commutes with χ_2 iff $(k_{f_i}^\times)^{d/e_i}$ commutes with $\alpha\eta^{-l}$. This is impossible if $d/e_i \nmid l$. (We need $\gamma\alpha = \alpha\gamma^{\sigma^{-l}}$ for all $\gamma \in (k_{f_i}^\times)^{d/e_i}$, hence all $\gamma \in K_{f_i}^{d/e_i}$. Let γ have a 1 in the jth entry and 0's elsewhere; if $\alpha = (\alpha_0, \dots, \alpha_{d-1})$, then $\gamma\alpha = (0, \dots, 0, \alpha_j, 0, \dots, 0)$, while $\alpha\gamma^{\sigma^{-l}}$ has a 0 as its jth entry if $d/e_i \nmid l$. So $\alpha_j = 0$ for all j. See Lemma 4.1 for a similar argument.) In the rest of the proof for this case, we use only this property of χ_1.

As noted above, it now suffices to show that $W_{(j)}$ commutes with χ_1. For this, it suffices to prove that $\chi_1^b(y) = \chi_1(y)$ for power-permutation matrices b and (e_i, f_i)-pure elements y such that both sides are defined, since such elements y generate H_l. Because b normalizes $(k_{f_i}^\times)^{d/e_i}$, it is easy to see that $b^{-1}H_lb \cap H_l$ is normalized by $(k_{f_i}^\times)^{d/e_i}$ and that χ_0^g is fixed by $(k_{f_i}^\times)^{d/e_i}$ on this group. To see that $\chi_0^b = \chi_0$ there, it suffices to show that χ has exactly one $(k_{f_i}^\times)^{d/e_i}$-stable extension from $b^{-1}H_{l+1}b \cap H_{l+1}$ to $b^{-1}H_lb \cap H_l$, since χ_1 is already known to be one such extension. So let χ_1 be any $(k_{f_i}^\times)^{d/e_i}$-stable extension of $\chi^b\chi^{-1}$ $(\equiv 1)$ to $b^{-1}H_lb \cap H_l$; we will prove that it is trivial.

This step, too, is done by induction: we assume that any such extension is trivial on $b^{-1}H_lb \cap H_l \cap K_d^r$ $(r \geq l)$ and prove it trivial on $b^{-1}H_lb \cap H_l \cap K_d^{r-1}$. Suppose that $y \in b^{-1}H_lb \cap H_l \cap K_d^r$, but $y \notin b^{-1}H_{l+1}b \cap H_{l+1}$. Then either y or byb^{-1} is in H_l but not in H_{l+1}; assume the former for definiteness. As noted above, we may assume that y is (e_i, f_i)-pure. If $y \in K_d^r$, the inductive hypothesis says that $\chi^b\chi^{-1}(y) = 1$. We show next that $(r-1)$ is not divisible by d/e_i. If $(r-1)$ is divisible by d/e_i, then y and byb^{-1} both lie in K_d^{r-1} but not in K_d^r, since η^{r-1} is central in $G_{(j)}$ and if $b \in W_{(j-1)}$ and $0 \neq \alpha \in m_d$, then $g\alpha g^{-1} \in A_d$ and $\notin A_d^1$. Since $y \in H_l$ and $y \notin H_{l+1}$, $l = r-1$. But by hypothesis, l is not divisible by d/e_i. This means that if $d/e_i|(r-1)$, then $b^{-1}H_lb \cap H_l \cap K_d^r = b^{-1}H_lb \cap H_l \cap K_d^{r-1}$, and the induction extends.

So we may assume that $(r - \mathrm{i})$ is not divisible by d/e_i. Let $y = 1 + y_0$, $y_0 = \gamma\eta^{j-1}$, be a new addition to the domain of χ_1 such that y is (e_i, f_i)-pure and the nonzero entries are $\equiv c \bmod n_i$ $(n_i = n/e_if_i = d/e_i)$. Assume for convenience that char $k \neq 2$ (the modifications for char $k = 2$ are easy). Let $\zeta \in m_e^{f_i}(e_i)^\times$ have I's in every entry except those $\equiv c \bmod n_i$, and $2I$'s in those. Then $\zeta y\zeta^{-1} = 1 + 2y_0$, and

$$\chi_1(\zeta y\zeta^{-1}y^{-1}) = 1, \quad \zeta y\zeta^{-1}y^{-1} \equiv y \bmod K_1^r.$$

Since $\chi_1(\zeta y\zeta^{-1}y^{-1}) = \chi_1(y)$ by the inductive hypothesis, $\chi_1(y) = 1$. This extends the induction and proves the result (with $\chi_0 = \chi_1$) when $d/e_i \nmid l$.

3a. Assume that $d/e_i|l$. Then, as noted above, the elements of H_l/H_{l+1} can be picked to be stable under the power-permutation matrices $W_{(j)}$ for $G_{(j)}$. We change notation and let $\chi_\alpha(1+y) = \psi \circ \mathrm{Tr}(\alpha\eta_{(j)}^{1-j}y)$ for $y \in K_e^{j-1}$. Since $\eta_{(j)}^{1-j} = \varepsilon_0\varpi^{1-j} + \cdots$, $\varepsilon_0 \in k_{f_i}^{d/e_i}$ (from 6.1(2)), the argument at the start of step 2 shows that $(k_{f_i}^\times)^{d/e_i}$ commutes with χ_α iff $\alpha \in k_{f_i}^{d/e_i}$. For $\gamma \in k_{f_i}$ and $y = \gamma\eta^{j-1}$, Lemma 4.8 gives

$$\chi_1^\eta(1+y)\chi_1^{-1}(1+y) = \chi_1(1+\eta y\eta^{-1})\chi_1(1+y)^{-1} = 1,$$

because η and y commute. Since $k_{f_i}^\times$ fixes χ_1^η, it fixes $\chi_1^\eta\chi_1^{-1}$; therefore

$$\chi_1^\eta = \chi_1\chi_\beta, \quad \beta \in k_{f_i}^e \quad \text{and} \quad \beta \perp k_{f_i} \text{ under } (\alpha, \beta) = \mathrm{Tr}\, \alpha\beta.$$

We also know that there are q^{df_i} characters χ_α of H_l/H_{l+1}. They form a commutative group on which η acts by conjugation. The fixed elements are the χ_α with $\alpha \in k_{f_i}$ (i.e., all components equal), since $\chi_\alpha^\eta = \chi_{\alpha'}$ with $\alpha'^\sigma = \alpha$. Thus there are $q^{f_i(d-1)}$ characters $\chi_\alpha^\eta\chi_\alpha^{-1}$. All of these annihilate the elements

$1+\gamma\eta^{j-1}$, $\gamma \in k_{f_i}$, since

$$\chi_\alpha^\eta \chi_\alpha^{-1}(1+\gamma\eta^{j-1}) = \psi \circ \operatorname{Tr} \alpha(\gamma^\sigma - \gamma)^{\sigma^{1-j}},$$

and $\gamma^\sigma = \gamma$ if $\gamma \in k_{f_i}$. Hence the $\chi_\alpha^\eta\chi_\alpha^{-1}$ exhaust the characters on H_l annihilating H_{l+1} and the $1+\gamma\eta^{j-1}$, $\gamma \in k_{f_i}$. Since $\chi_{-\beta}$ is such a character, $\chi_\alpha^\eta\chi_\alpha^{-1} = \chi_{-\beta}$ for some α. Set $\chi_0 = \chi_1\chi_\alpha$. Then χ_0 commutes with η and with $(k_{f_i}^\times)^{d/e_i}$. (These properties of χ_0 are what we use in further analyses of this case.)

It now suffices to show that b commutes with χ_0 whenever $b \in W_{(j)}$. Elements of $W_{(j)}$ take elements $\gamma\eta_{(j)}^{j-1}$, $\gamma \in m_e$, to elements $\equiv \gamma'\eta_{(j)}^{j-1} \bmod H_{l+d}$, as noted above. It follows easily that if $y \in H_l$ and $byb^{-1} \in K_e^j$, then $y \in H_{l+1}$. We therefore need only prove that $\chi_0(y) = \chi_0(byb^{-1})$ for our coset representatives $y = 1+\gamma\eta^l$. It suffices to check this when γ has only one nonzero entry, at (say) the cth place. It also suffices to assume that b is a permutation matrix in $G_{(j)}$ or a "diagonal" matrix,

$$PgP^{-1} = \begin{bmatrix} \gamma_0\xi^{a_0} & & 0 \\ & \ddots & \\ 0 & & \gamma_{n_i-1}\xi^{a_{n_i-1}} \end{bmatrix},$$

P the permutation matrix as in §2, $\gamma_h \in k_{f_i}$, where ξ is defined by saying that $P\eta^{n_i}P^{-1}$ is a "diagonal" matrix of ξ's. (Recall: $n_i = n/e_if_i$. Each entry is an $e_i \times e_i$ block matrix whose $f_i \times f_i$ blocks are in k_{f_i}; the blocks corresponding to ξ^{a_l} have indices $\equiv l \bmod n_i$.) For "diagonal" b, let $l \equiv c \bmod n_i$. Then

$$\chi_0(gyg^{-1}) = \chi_0(\gamma_h\eta^{d_ia_h} \cdot y(\gamma_h\eta^{d_ia_h})^{-1}) = \chi_0(y),$$

which takes care of this case.

Here is a brief illustration of this last point. For the matrix

$$\eta = \eta_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \varpi & 0 & \varpi & 0 \\ 0 & \varpi & 0 & \varpi \end{bmatrix}$$

used in the example for Lemma 3.3, we have

$$\xi = \begin{bmatrix} 0 & 1 \\ \varpi & \varpi \end{bmatrix}.$$

The "diagonal" matrix corresponding to $\left[\begin{smallmatrix} \xi^4 & 0 \\ 0 & \xi \end{smallmatrix}\right]$ is

$$b_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & a\varpi^2 & 0 & b\varpi^2 \\ \varpi & 0 & \varpi & 0 \\ 0 & b\varpi^3 & 0 & c\varpi^2 \end{bmatrix}, \qquad a, b, c \text{ as in §3}.$$

It is easy to see that for $1+\gamma\eta^{2h}$ with $\gamma = (0, \gamma_1, 0, \gamma_3)$, so that the matrix for $\gamma\eta^{2j}$ has entries only in the starred positions of

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & * \end{bmatrix},$$

we have $b_1(1+\gamma\eta^{2h})b_1^{-1} = \eta^4(1+\gamma\eta^{2h})\eta^{-4}$, since b_1 and η^4 have the same entries in the rows and columns with the starred entries.

3b. The permutation matrices in $G_{(j)}$ are (e_i, f_i)-permutation matrices and normalize $(k_{f_i}^\times)^d$. So if b is one of these permutation matrices, then $(k_{f_i}^\times)^d$ also fixes χ_0^b and must therefore be of the form $\chi_0\chi_\alpha$, $\alpha \in (k_{f_i}^\times)^d$. The map $\varphi\colon b \mapsto \alpha$ (which we regard as a map from $\mathscr{S}_{n_i}$ to $(k_{f_i})^d$; $\mathscr{S}_{n_i}$ acts on each block of n_i symbols in the same way) has the following properties:

(i) φ is a 1-cocycle: $\varphi(uv) = \varphi(u)^v + \varphi(v)$;

(ii) $\varphi(u) = 0$ if u is the cyclic transposition $(0, 1, \ldots, n_i - 1)$ (then u is the product of η and a "diagonal" element);

(iii) $\varphi(v_1) = (\alpha_0, \ldots, \alpha_{d-1})$ has $\alpha_b = 0$ if b is fixed by v_1. (We then have $v_1 y v_1^{-1} = y$ if y is (e_i, f_i)-pure with nonzero entries only at indices $\equiv b \bmod n_i$.)

It is clear that (i) and (iii) imply

(iv) If $\varphi(v_2) = (\beta_0, \ldots, \beta_{d-1})$ and v_1, v_2 both take b to c, then $\alpha_b = \beta_b$. (Write $v_2 = (v_2v_1^{-1})v_1$.)

The permutation v of order 2 given by

$$(0, n_i-1)(1, n_i-2)\cdots\left(\frac{n_i-2}{2}, \frac{n_i}{2}\right) \quad \text{if } n_i \text{ is even}$$

and

$$(0, n_i-1)(1, n_i-2)\cdots\left(\frac{n_i-3}{2}, \frac{n_i+1}{2}\right) \quad \text{if } n_i \text{ is odd}$$

satisfies $vuv = u^{-1}$, where u is as in (ii). Therefore $\varphi(v)^u = \varphi(v)$, which implies that $\varphi(v) = \beta \in k_{f_i}$.

Suppose first that $p \neq 2$. Since v has order 2 and $\varphi(v)^v = \varphi(v)$, $2\beta = 0$ or $\beta = 0$. From(iv), $\varphi((0, n_i-1)) = 0$. Conjugating by u, we get $\varphi((b, b+1)) = 0$ for all b. It follows that $\varphi \equiv 0$.

If $p = 2$, we work a bit harder. If n is odd, then $\beta = 0$ because $(n_i-1)/2$ is fixed by v. We get $\varphi \equiv 0$ as in the case $p \neq 2$. If n is even, then we have $\varphi((0, n_i-1)) = (\beta, 0, 0, \ldots, 0, \beta)$, and conjugation with u gives $\varphi((b, b+1)) = (0, 0, \ldots, 0, \beta, \beta, 0, \ldots, 0)$, with the β's in the b, $b+1$ places. We now have

$$\varphi((0,1,2)) = \varphi((1,2))^{(0,1)} + \varphi((0,1)) = \varphi((1,2)),$$

$$\begin{aligned}\varphi((0,1,2,3)) &= \varphi((2,3)(0,1,2)) = \varphi((2,3))^{(0,1,2)} + \varphi((0,1,2))\\ &= \varphi((0,1)) + \varphi((2,3)) \text{ (as one sees by calculating the terms)},\end{aligned}$$

$$\varphi((0,1,2,3,4)) = \varphi((0,1)) + \varphi((3,4)),$$

$$\vdots$$

$$\varphi((0,1,\ldots,n_i-1)) = \varphi((0,1)) + \varphi((2,3)) + \cdots + \varphi((n_i-2, n_i-1)) = \beta.$$

Hence $\beta = 0$, and we finish as before. □

For the rest of the proof of Theorem 6.1, we make the notational convention that if χ is the given representation on K_e^j, then χ_0 is a (fixed) extension of χ to K_e^{j-1} with the property of the lemma. (We shall then construct an extension of χ to K_e^{j-1}, which we also call χ; often $\chi \neq \chi_0$.)

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We return to the proof of Theorem 6.1. Assume that we have defined χ on H_0^j, that $t'_{h-1} \geq j > t'_h$, and that i is the largest index with $j \leq t_i$. We need to extend χ to H_0^{j-1}; the procedure is slightly different in each of the three remaining steps, which we also think of as different cases. The hard part is generally in finding $\eta_{(j-1)}$; in constructing it, we usually verify most of (1)–(11).

(a) (*Step* 3). Assume that $j-1 > t_{i+1}$ and $e \nmid (j-1)e_i$. Then set $\chi = \chi_0$ and $\eta_{(j-1)} = \eta_{(j)}$. It follows that $E_{(j-1)} = E_{(j)}$, etc.; all of (1)–(11) are vacuous or trivial except for (7), which we now treat. (Occasionally the treatment that follows is slightly more complicated than necessary; this is to ensure that essentially the same proof applies to the other cases). Note that $C_{j-1} = 1$.

It is easy enough to check that $K_e^{t_c+2-j} \cap G_{c-1}$ commutes with $\chi|_{H_0^{j-1}}$ when $h \leq c \leq i$; one applies Lemma 4.2. (For instance, if $x \in K_e^{t_i+2-j} \cap G_{i-1}$ and $x-1$ is invertible, then $\chi^x = \chi$ on $K_e^{t_i+1}$, while if $y \in H_0^{j-1}$ and $y-1$ is invertible mod H_0^j, then $\chi^y = \chi$ on $K_e^{t_i+1}$.) The proof that $K_e^{t''_c} \cap G_{c-1}$ commutes with $\chi|_{H_0^{j-1}}$ when $0 \leq c < h$ $(G_0 = G)$ is the same. We have constructed χ $(= \chi_0)$ so that $G_{(j-1)}$ $(= G_{(j)})$ commutes with χ on H_0^{j-1}. Conversely, suppose that $w \in Z_e K_e$ commutes with χ. Then w commutes with $\chi|_{H_0^j}$, and (7) applied to $\chi|_{H_0^j}$ implies that we can write

$$\begin{aligned} w &= \beta_0 \eta_{(j)}^{j_0}(1 + \beta_1\eta_{(j)} + \cdots + \beta_{t_i-j}\eta_{(j)}^{t_i-j} + \beta_{t_i-j+1}\eta_{i-1}^{t_i-j+1} + \cdots) \\ &= \beta_0 \eta_{(j-1)}^{j_0}(1 + \beta_1\eta_{(j-1)} + \cdots + \beta_{t_i-j}\eta_{(j-1)}^{t_i-j} + \beta_{t_i-j+1}\eta_{i-1}^{t_i-j+1} + \cdots), \end{aligned} \tag{7.1}$$

with $\beta_g \in m_e$ for all g, $\beta_0 \in m_e^\times$, and $\beta_g \in m_e^{f_i}(e_i)$ for $g \leq t_i - j$, $\beta_g \in m_e^{f_{i-1}}(e_{i-1})$ for $t_i - j < g \leq t_{i-1} - j$, etc. We need to prove first that $\bmod A_e^{t_i-j+2}$, $\beta_{t_i-j+1}\eta_{i-1}^{t_i-j+1} \equiv$ some $u \in M_{(j-1)}$. Dividing by an element known to commute with χ and writing β for β_{t_i-j+1}, we may assume that $w = 1 + \beta\eta_{i-1}^{t_i-j+1} + \cdots = w_1 w_0$, where $w_0 = 1 + \beta\eta_{i-1}^{t_i-j+1}$.

We show first that $\chi^{w_1}(y) = \chi(y)$ if $y = 1 + \gamma\eta_{i-1}^{j-1}$ with $\gamma \in m_e^{f_{i-1}}(e_{i-1})$. Note that if $g \leq i-1$, then $\eta_{i-1} \equiv \zeta\eta_g$ for some $\zeta \in k_{f_i}^\times$, by an induction using (5). Because ζ and γ are both in $m_e^{f_g}(e_g)$, it follows that y is congruent mod K_e^j to an element of G_g. In fact, since y and ζ_g are both in H_0, this congruence also holds mod H_0^j.

Since $w_1 \in (K_e^{t_i-j+1} \cap G_{i-1})(K_e^{t_i-j+1} \cap G_{i-2}) \cdots (K_e^{t_h-j+2} \cap G_{h-1})$, it suffices to prove that $\chi^{w_1}(y) = \chi(y)$ if w_1 is in any one of the factors. If $w_1 \in (K_e^{t_i-j+1} \cap G_{i-1})$, then $\chi^{w_1}(y) = \chi(y)$ by Lemma 4.2 (since G_{i-1} commutes with χ on $H_0^{t_i+1}$). If $w_1 \in (K_e^{t_{i-1}-j+1} \cap G_{i-2})$, then $(w_1, y) \in H_0^{t_{i-1}}$, and we are concerned with χ on that group. Write $\chi = \chi_{0,i-1}\chi_{1,i-1}$, as in (8). Then $\chi_{0,i-1}((w_1, y)) = 1$ by Lemma 4.2, since G_{i-1} commutes with $\chi_{1,i-1}$, while $\chi_{1,i-1}((w_1, y)) = 1$ by Lemma 4.7, since y is congruent mod H_0^j to an element of G_{i-1}. This proves the claim for $w_1 \in (K_e^{t_{i-1}-j+1} \cap G_{i-2})$, and the proof for the other factors is nearly the same.

Therefore $\chi^w \cdot \chi^{-1}(y) = \chi(w_0 y w_0^{-1} y^{-1})$ and $w_0 y w_0^{-1} y^{-1} \in H_0^{t_i}$. Write

$\chi|_{H_0^{t_i}} = \chi_{0,i}\chi_{1,i}$, as in (8). Since $G_{(i-1)}$ commutes with $\chi_{0,i}$,

$$\chi_{0,i}(w_0 y w_0^{-1} y^{-1}) = 1 .$$

By (8) and Lemma 4.7, $\chi_{1,i}(w_0 y w_0^{-1} y^{-1}) = 1$ for all y only if w_0 is congruent mod $K_e^{t_i-j+2}$ to an element w_0' of $G_{\widetilde{(t_i)}}$, the group of invertible elements in the algebra $M_{\widetilde{(t_i)}}$ defined in (8). (Note that y is congruent modulo H_0^j to an element of G_{i-1}.) By (iv) of (8), together with (5), $w_0' \equiv w' \bmod K_e^{t_i-j+2}$, where $w' \in G_{(j)} \cap K_e^{t_i-j+1}$. We already know that $\chi^{w'} = \chi$.

Dividing by w', we may write

$$w = 1 + \beta_{t_i-j+2}\eta_{i-1}^{t_i-j+2} + \cdots + \beta_{t_{i-1}-j}\eta_{i-1}^{t_{i-1}-j} + \beta_{t_{i-1}-j+1}\eta_{i-2}^{t_{i-1}-j+1} + \cdots$$

(where the β_l may be different from what they were in the previous expression for w). The inductive hypothesis lets us write $w = 1 + \beta_{t_{i-1}-j+1}\eta_{i-2}^{t_{i-1}-j+1} + \cdots$, $\beta_{t_{i-1}-j+1} \in m_e^{e_{i-1}}(f_{i-1})$, etc.; an argument like the one above shows that w is congruent modulo $K_e^{t_{i-1}-j+2}$ to an element in G_{i-1}. We continue inductively to get (7). This concludes the proof in case (a).

(b) (*Step* 4). Assume that $j-1 > t_{i+1}$ and $e|(j-1)e_i$. Set

$$\chi(1+y) = \chi_0(1+y)\psi \circ \mathrm{Tr}(\alpha'\eta_{(j)}^{-j+1}y) = \chi_0(1+y)\chi_{\alpha'}(1+y),$$

where $\alpha' \in m_e$ is such that $\alpha = \mathrm{Tr}_{e_i}\alpha' \in k_{f_i}$. (Note that $y = \gamma\eta_{(j)}^{j-1}$ for some $\gamma \in m_e^{f_{h-1}}(e_{h-1})$. There is therefore some ambiguity in α'; what determines χ is not α' but $\mathrm{Tr}_{e_{h-1}}\alpha'$. We could find an $\alpha'' \in m_e^{f_{h-1}}(e_{h-1})$ that gives the character, but then we would need to take traces over F_{h-1}. There seems to be no advantage in introducing the additional notation.) We choose one α' for each α, with the added requirements that each component of α' is in k_{f_i}; that the first e_i components of α' are all the same, the second e_i components are all the same, etc.; and that 0 corresponds to 0. For instance, we might make all components of α' beyond the first e_i equal to 0. (If $\alpha' = 0$, then of course $\chi = \chi_0$.) This gives us $q^{f_i} = C_{j-1}$ choices for χ. (It is in fact the case that if $\alpha = 0$, then χ and χ_0 and conjugate regardless of what α' is chosen, but we shall not need this fact here. It is proved by applying Lemma 4.7 repeatedly.) We shall show below that these extensions (for different α) are nonconjugate under Z_eK_e. A computation in the proof of Lemma 6.2, case (2), shows that if $\gamma \in (k_{f_i}^e)^\times$, then $\chi^\gamma = \chi$; similarly, χ is fixed by all (e_i, f_i)-permutation matrices because α' is.

We next produce $\eta_{(j-1)}$ and $E_{(j-1)}$. This is a complicated affair because we need to insure that $E_{(j-1)}$ is nicely embedded. An example may help to explain the problem that may arise. Let $n = e = 4$, $f = 1$, $e_1 = 2$, $s_1 = t_1 = 10$, and $s_2 = t_2 < 8$. Suppose that for $\varpi = \varpi_4$ and $\gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)$, $\gamma_j \in k$, we have

$$\chi(1+\gamma\varpi^{10}) = \psi\left(\sum_{j=0}^{3}\gamma_j\right).$$

Then we may take $E_{(10)} = F[\varpi^2]$ and $\eta_{(10)} = \varpi$. From case (a), $E_{(9)} = E_{(10)}$. For level 8, we produce χ_0 and set $\chi = \chi_0\chi_1$, where

$$\chi_1(1+\gamma\varpi^8) = \psi(\gamma_0+\gamma_1) \quad \text{if } \gamma\varpi^8 \in G_{(9)}.$$

This last condition means that $\gamma = (\gamma_0, \gamma_1, \gamma_0, \gamma_1)$. Then $\chi_1 = \psi \circ \mathrm{Tr}^{(2)}(\alpha\gamma)$, where we may take $\alpha = (1, 1, 1, 1) \in m_4^1(2)$. We need α'. If $p \neq 2$, we can take $\alpha' = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. It is then clear that $m_4^1(2)$ and ϖ commute with χ_1, so that $E_{(8)} = E_{(10)}$. (This is one reason that the tamely ramified case is easier than the general case.) But if $p = 2$, this choice is impossible. We may take $\alpha' = (1, 1, 0, 0)$; recall that the first two entries of α' must be equal. Now (since char $k = 2$, so that $+$ and $-$ are the same)

$$\chi_1^{\varpi}\chi_1^{-1}(1 + \gamma\varpi^8) = \psi\left(\sum_{j=0}^{3} \gamma_j\right), \qquad \gamma \in m_4,$$

as a calculation shows. But for $\delta = (\delta_0, \delta_1, \delta_2, \delta_3)$ and $w = 1 + \delta\varpi^2$,

$$\chi_0^{w}\chi_0^{-1}(1 + \gamma\varpi^8) = \psi\left(\sum_{j=0}^{3} \gamma_j(\delta_j + \delta_{j+2})\right)$$

(extend δ_j periodically with period 4). So $\chi^{\varpi w} = \chi$ if $\delta_j + \delta_{j+2} = 1$, so that $\delta = (\delta_0, \delta_1, \delta_0 + 1, \delta_1 + 1)$. If we take $\delta_0 = \delta_1 = 1$, then $F[(\varpi w)^2]$ is nicely embedded and gives $E_{(8)}$. But if not, $F[(\varpi w)^2]$ need not be nicely embedded. The following is one way of forcing the nice embedding of the field.

We begin not with $\eta_{(j-1)}$, but with something closer to $\eta_{(j-1)}^{e/e_i}$. Let $\theta_{(j)} = \eta_{(j)}^{e/e_i}$. We set

$$\theta_{(j-1)} = \theta_{(j)}(1 + \delta_{i-1}\eta_{i-1}^{t_i - j}) \cdots (1 + \delta_{h-1}\eta_{h-1}^{t_h - j}),$$

where $\delta_g \in k_{f_g}^e$, $\delta_g^{\sigma^{e/e_i}} = \delta_g$, and the δ_g are chosen so that $\theta_{(j-1)}$ commutes with χ. We choose the δ_g inductively, as follows: $\chi^{\theta_{(j)}}\chi^{-1}$ is trivial on $H_0^j(H_0^{j-1} \cap G_{(j)})$, and by (9) we can choose δ_{i-1} so that if $w_1 = \theta_{(j)}(1 + \delta_{i-1}\eta_{i-1}^{t_i - j})$, then $\chi^{w_1}\chi^{-1}$ is trivial on $H_0^j(H_0^{j-1} \cap G_{i-1})$. Now (9) lets us choose δ_{i-2} so that if $w_2 = w_1(1 + \delta_{i-2}\eta_{i-2}^{t_{i-1} - j})$, then $\chi^{w_2}\chi^{-1}$ is trivial on $H_0^j(H_0^{j-1} \cap G_{i-2})$. Continue.

The matrix θ has a special and useful form. Regard elements of M as $n/f_i \times n/f_i$ block matrices (with $f_i \times f_i$ blocks), and let P be the block permutation matrix taking $(0, 1, \ldots, n/f_i - 1)$ to $(0, n_i, \ldots, (e_i - 1)n_i, 1, n_i + 1, \ldots, n_i - 1, \ldots, n/f_i - 1)$, as in §2. Then each term $P^{-1}\theta_{(j)}P$, $P^{-1}(1 + \delta_{i-1}\eta_{i-1}^{t_i - j}P, \ldots, P^{-1}(1 + \delta_{h-1}\eta_{h-1}^{t_h - 1})P$ is a "diagonal" block matrix consisting of n_i blocks, each block an $e_i \times e_i$ block matrix of ($f_i \times f_i$ blocks that are) elements of F_{f_i}. The same statement therefore holds for $P^{-1}\theta_{(j-1)}P = \xi$ (say). Let the blocks for ξ be $(\xi_0, \ldots, \xi_{n_i - 1})$. Similarly, set $P^{-1}\theta_{(j)}P = \xi' = (\xi'_0, \ldots, \xi'_0)$ and $P^{-1}(1 + \delta_g\eta_g^{t_{g+1} - j})P = \tau_g = (\tau_{g,0}, \ldots, \tau_{g,n_i - 1})$. (Since $\theta_{(j)} \in E_{(j)}$ and $E_{(j)}$ is nicely embedded, the entries for ξ' are all the same.) We then have

$$\xi_l = \xi'\tau_{i-1,l} \cdots \tau_{h-1,l}, \qquad 0 \leq l \leq n_i - 1.$$

If we now write $\mu_g = (\tau_{g,0}, 1, \ldots, 1)$, then $P\mu_g P^{-1}$ commutes with $\chi|_{H^j}$. (For if $P^{-1}\delta_g P = (\delta_{g,0}, \ldots, \delta_{g,n_i - 1})$, then $P\mu_g P^{-1} = (1 + \delta'_g\eta_g^{t_{g+i} - j})$, where $P^{-1}\delta'_g P = (\delta_{g,0}, 0, \ldots, 0)$; thus $\delta'_g \in m_e^{f_g}(e_g)$. Now use property (7).) Since

$P(\xi', 1, \dots, 1)P^{-1}$ also commutes with $\chi|_{H^j}$, we see that $\kappa = P(\xi_0, 1, \dots, 1)P^{-1}$ commutes with $\chi|_{H^j}$. We can prove that κ commutes with χ on H^{j-1} by showing that $\chi^\kappa(y) = \chi(y)$ for a set of generators y of the group H^{j-1}/H^j. We choose the y to be (e_i, f_i)-pure. Then $\kappa y \kappa^{-1} = \theta_{(j-1)} y \theta_{(j-1)}^{-1}$ if the entries of $y-1$ (as a block matrix with $f_i \times f_i$ blocks) are in rows and columns with indices divisible by $n/e_i f_i$, and $\kappa y \kappa^{-1} = y$ otherwise. In either case, $\chi^\kappa(y) = \chi(y)$.

We are now almost done with the construction of $\eta_{(j-1)}$. Let λ be the $n_i \times n_i$ block matrix

$$\lambda = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & I \\ \xi_0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and let $\zeta_{(j-1)} = P\lambda P^{-1}$, $\eta_{(j-1)} = \zeta_{(j-1)}^{f/f_i}$. Since λ^{n_i} is the $n_i \times n_i$ block matrix $(\xi_0, \dots, \xi_0)$, $F_{f_i}[\zeta_{(j-1)}^{n_i}] = E_{(j-1)}$ is nicely embedded. (Note: $n_i = ef/e_i f_i$.) We also have

$$\eta_{(j-1)} = \eta_{(j)} P^{-1} \mu_{i-1} \cdots \mu_{h-1} P = \eta_{(j)}(1 + \delta'_{i-1}\eta_{i-1}^{t_i - j}) \cdots (1 + \delta'_{h-1}\eta_{h-1}^{t_h - j});$$

since each $(1 + \delta'_g \eta_g^{t_{g+1}-j})$ commutes with $m_e^{f_i}(e_i)$, conjugation by $\eta_{(j-1)}$ acts as σ on $m_e^{f_i}(e_i)$. Furthermore, Lemma 3.3 applies to $E_{(j)}$, $E_{(j-1)}$, so that the power-permutation matrices for $G_{(j-1)} = \mathrm{GL}_{n_i}(E_{(j-1)})$ all commute with $\chi|_{H_0^j}$. Thus it suffices to prove that for any power-permutation matrix b, $\chi^b(y) = \chi(y)$ when y is a set of (e_i, f_i)-pure generators of H_0^{j-1}/H_0^j, as before. Since the elements byb^{-1} are then also a set of (e_i, f_i)-pure generators of H_0^{j-1}/H_0^j, it suffices to consider a set of b generating the power-permutation matrices, for instance κ and the (e_i, f_i)-permutation matrices. But we have already verified that $\chi^b(y) = \chi(y)$ for these elements.

In the course of constructing $\eta_{(j-1)}$ and $E_{(j-1)}$, we have verified properties (1)–(3), (5), and half of (7). Furthermore, (4) is clear, (8)–(10) follow directly from the inductive hypothesis, and the other part of (7) is proved almost exactly as in case (a). (Note that (7.1) holds because of (5) and Lemma 3.1; the β_l for $l \geq t_i - j + 1$ may change from one line to the next.) So we need only verify (11), that different choices for α' give nonconjugate extensions of χ. Consider two extensions $\chi = \chi_0 \cdot \chi_{\alpha'}$, $\chi_0 \cdot \chi_{\beta'}$, where, e.g., $\chi_{\alpha'}(1+y) = \psi \circ \mathrm{Tr}(\alpha' \eta_{(j)}^{-j+1} y)$, $y \in K_e^{j-1}$. Suppose that these are conjugate by $x \in Z_e K_e$ and that $\alpha' \neq \beta'$. Then x commutes with $\chi_0|_{K_e^j}$, but not with $\chi_0|_{K_e^{j-1}}$, and (7) (for j, $j-1$) shows that we may take $x = (1 + \delta_{i-1}\eta_{i-1}^{t_i - j+1}) \cdots (1 + \delta_h \eta_h^{t_{h+1}-j+1})$ with $\delta_g \in m^{f_g}(e_g)$. We show that $\chi^x(y) = \chi(y)$ for all $y = 1 + \gamma \eta_{(j-1)}^{j-1}$ with $\gamma \in m_e^{f_i}(e_i)$. The proof is like a part of that used in (7). We may work with one factor of x at a time; thus we assume first that $x = 1 + \delta_{i-1}\eta_{i-1}^{t_i - j+1}$. Then $(x, y) \in H_0^{t_i}$. Write χ on $H_0^{t_i}$ as $\chi_{0,i}\chi_{1,i}$, as in (8). Since $x \in G_{i-1}$, Lemma 4.2(b) implies that $\chi_{0,i}((x, y)) = 1$; since y is congruent mod H_0^j to an element of $G_{(t_i)}$ (by the same reasoning as in (a)), (4.7) implies that $\chi_{1,i}((x, y)) = 1$. Thus

$\chi^x(y) = \chi(y)$ in this case, and the same argument (with only indices altered) applies to the other factors.

Therefore $\chi^x = \chi$ on $G_{(j-1)} \cap K_e^j$, so that $\mathrm{Tr}_{e_i}(\beta') = \mathrm{Tr}_{e_i}(\alpha')$. By construction, $\alpha' = \beta'$, contradicting our assumption that $\alpha' \neq \beta'$. This concludes the proof in case (b).

(c) (Step 5) Assume that $j-1 = t_{i+1}$; then $e|(j-1)e_{i+1}$. We let $\eta_i = \eta_{(j)}$, $E_i = E_{(j)}$, and $M_i = M_{(j)}$; we set

$$\chi(1+y) = \chi_0(1+y)\psi \circ \mathrm{Tr}(\alpha'\eta_i^{1-j}y) = \chi_0(1+y)\chi'(1+y),$$

say, where α' has the property that if $\alpha_{i+1} = \mathrm{Tr}_{e_i}\alpha'$, then $E_i[\alpha_{i+1}\eta_i^{1-j}] = E_{(j-1)}^{\sim} = E_{(t_{i+1})}^{\sim}$ has ramification index e_{i+1}/e_i and residue class degree f_{i+1}/f_i over E_i. (The condition on the ramification index follows from (1.3).) We also assume that all entries of α' are in $k_{f_{i+1}}$. An argument like that for the case of t_1 shows that we may assume that the first e/e_{i+1} entries of α' are all the same, the next e/e_{i+1} are all the same, and so on. This makes $E_{(j-1)}^{\sim}$ nicely embedded. Then χ' commutes with all (e_{i+1}, f_{i+1})- permutation matrices, and hence χ does as well. This proves part of the second part of (7). We define $\chi_{0,i+1} = \chi_0$, $\chi_{1,i+1} = \chi'$ in (8). Choose one $\alpha\eta_i^{1-j}$ from each Z_eK_e-conjugacy class and one α' for each α. Write $f_{i+1} = f'_{i+1}f_i$.

Suppose that $w = \delta_0\eta_i^h(1+\delta_1\eta_i+\cdots) = w_1w_2 \in Z_eK_e$ commutes with χ. We show that $w_1 = \delta_0\eta_i^h$ commutes with $E_{(t_{i+1})}^{\sim}$. We may assume that w commutes with $\chi|_{K_e^j}$, so that $\delta_0\eta_i^h$ commutes with E_i, from (7); hence we need only prove that $\delta_0\eta_i^h$ commutes with $\alpha\eta_i^{1-j}$. Write $y = 1+\gamma\eta_i^{j-1} = 1+y_1$, with $\gamma \in m_e^{f_i}(e_i)$. Then

$$\chi(wyw^{-1}y^{-1}) = \chi_0(wyw^{-1}y^{-1})\chi'(wyw^{-1}y^{-1}).$$

But $\chi_0(wyw^{-1}y^{-1}) = 1$ because $w \in G_i$, and $\chi'^w = \chi'^{w_1}$ because commutators (y, w_2) are in $K_e^{t_i+1}$. Now Lemma 4.8 shows that w_1 commutes with $\alpha\eta_i^{1-j}$. Note that for $h=0$, this means that $\delta_0 \in (m_e^{f_{i+1}}(e_{i+1}))^\times$; conversely, the calculation shows that $(m_e^{f_{i+1}}(e_{i+1}))^\times$ commutes with χ.

We next construct $\eta_{(j-1)}$. The procedure is like that used in (b), and we omit details when the calculations are essentially the same as in (b). We begin with $\theta_i = \eta_i^{e/e_i}$,

$$\theta_{(j-1)} = \delta_i\theta_i(1+\delta_{i-1}\eta_{i-1}^{t_i-t_{i+1}})(1+\delta_{i-2}\eta_{i-2}^{t_{i-1}-t_{i+1}})\cdots(1+\delta_{h-1}\eta_{h-1}^{t_h-t_{i+1}}),$$

where the $\delta_i \in k_{f_{i+1}}^e$ are to be determined. Let τ_i be a prime element in $F_i[\alpha\eta_i^{-t_{i+1}}]$ of the form $(\alpha\eta_i^{-t_i})^a\eta_i^{eb/e_i}$ (where $(e/e_i)b - t_ia = e/e_{i+1}$; this is possible by (1.3), since $n/f = e$ and $s_i/f = t_i$), chosen so that $F_i[\tau_i] = F_i[\alpha\eta_i^{-t_i+1}]$; this element is then of the form $\delta_i\theta_i$. The calculation given just before the start of this construction shows that $\chi^{\theta_i}\chi^{-1}(1+\gamma\eta_i^{t_{i+1}}) = 1 \quad \forall\gamma \in m_e^{f_i}(e_i)$. We now choose the δ_g inductively, using (8) and Lemma 4.7 as in case (b), to make $\chi^{\theta_{(j-1)}} = \chi$. Because the t_g, $g \leq i+1$, are all divisible by f_{i+1}, it is straightforward to check that the terms $\delta_g\eta^{t_g-t_{i+1}}$ all commute with $m_e^{f_{i+1}}(e_{i+1})$. Furthermore, if we regard elements of M as $n/f_{i+1} \times n/f_{i+1}$ block matrices and let P be the block permutation matrix taking $(0, 1, \ldots, n/f_{i+1}-1)$ to

$(0, n_{i+1}, \dots, (e_{i+1}-1)n_{i+1}, 1, n_{i+1}+1, 2n_{i+1}+1, \dots, n_{i+1}-1, \dots, n/f_{i+1}-1)$, as in §2 (with $n_{i+1}e_{i+1}f_{i+1} = n$), then $P^{-1}\theta_{(j-1)}P = \xi$ is a "diagonal" block matrix consisting of n_{i+1} blocks, each block an $e_{i+1} \times e_{i+1}$ block of elements of $F_{f_{i+1}}$ (embedded as $f_{i+1} \times f_{i+1}$ matrices). Let the blocks for ξ be $(\xi_0, \dots, \xi_{n_{i+1}-1})$; notice that $P^{-1}\tau_i P$ is of the form $(\xi_0', \xi_0', \dots)$. Just as in case (b), we show that for $\kappa = P(\xi_0, 1, \dots, 1)P^{-1}$, $\chi^\kappa = \chi$ on their common domain, and that if λ is the $n_{i+1} \times n_{i+1}$ block matrix

$$\begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & 0 & I \\ \xi_0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$\zeta_{(j-1)} = P\lambda P^{-1}$, $\eta_{(j-1)} = \zeta_{(j-1)}^{f/f_{i+1}}$, and $E_{(j-1)} = F_{f_{i+1}}[\zeta_{(j-1)}^{n_{i+1}}]$, then $E_{(j-1)}$ is nicely embedded and $\eta_{(j-1)}$ acts as σ on $m_e^{f_{i+1}}(e_{i+1})$. (Note that $n_{i+1} = e/e_{i+1} \cdot f/f_{i+1}$.) Since Lemma 3.3 applies to $E_{(j-1)}^{\sim}$ and $E_{(j-1)}$, the power-permutation matrices for $G_{(j-1)} = \mathrm{GL}_{n_{i+1}}^{n_{i+1}}(E_{(j-1)})$ all commute with $\chi|_{H_0^j}$. Therefore it suffices to prove that for any power-permutation matrix b of $G_{(j-1)}$, $\chi^b(y) = \chi(y)$ when y runs through a set of (e_{i+1}, f_{i+1})-pure elements generating H_0^{j-1}/H_0^j. Because $byb^{-1} \in H_0^{j-1}$ for all such b, y, it suffices to consider a set of x generating the power-permutation matrices, for instance κ and the (e_i, f_i)-permutation matrices. We have verified that $\chi^b(y) = \chi(y)$ for these matrices, so that $G_{(j-1)}$ commutes with χ; this is part of (7). We also set $\eta_{(t_i)}^{\sim} = (P\lambda' P^{-1})^{f/f_i+1}$, where

$$\lambda = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & 0 & I \\ \xi_0' & 0 & \cdots & \cdots & 0 \end{bmatrix};$$

then $(\eta_{(t_i)}^{\sim})^{e/e_{i+1}} = \tau_i$.

The construction shows, as in case (b), that (1)–(6) and the second part of (7) all hold; (8) is also clear. For the first part of (7), one inclusion goes just as in (b). To prove the other, suppose that w commutes with χ. Since w commutes with $\chi|_{K_e^j}$, we have

$$w = \delta_0' \eta_i^{h_0}(1 + \delta_1' \eta_1 + \cdots), \qquad \delta_0 \in m_e^{f_i}(e_i).$$

But $\delta_0' \eta_i^{h_0}$ must commute with $\alpha \eta_i^{-j}$, as we saw above; from (6) and (8), we can write

$$w = \varepsilon_0 \eta_{(j-1)}^{h_0}(1 + \varepsilon_1 \eta_i + \cdots), \qquad \varepsilon_0 \in (m_e^{f_{i+1}}(e_{i+1}))^\times.$$

Since $\varepsilon_0 \eta_{(j-1)}^{h_0}$ commutes with χ, we may delete it; the proof now goes as in (b).

That leaves (11). Let N_0^j, N_0^{j-1} be the groups of elements of $Z_e K_e$ commuting with χ on H_0^j, H_0^{j-1} respectively. If χ and $\chi^\# = \chi_0 \chi^\sim$ (where $\chi^\sim$ is given by β' just as χ' is given by α') are conjugate by an element $w \in Z_e K_e$, then $w \in N_0^j$ and w is determined mod N_0^{j-1}. So we may write

$$w = \delta_1 \eta_i^a (1 + \delta_{i-1} \eta_{i-1}^{t_i - t_{i+1}}) \cdots (1 + \delta_{h-1} \eta_{h-1}^{t_h - t_{i+1}}), \qquad \delta_g \in (k_{f_g})^e.$$

Write $\beta_{i+1} = \mathrm{Tr}_{e_i} \beta'$. Conjugation by $\delta_i \eta_i^a$ takes $\alpha_{i+1} \eta_i^{1-j}$ to $\beta_{i+1} \eta_i^{1-j}$ (since a calculation, using Lemma 4.7 and like some done in the course of verifying (7), shows that $(\delta_i \eta_i^a)^{-1} w$ fixes $\chi^\#$ on $K_e^{j-1} \cap G_i$). Conversely, if $\alpha_{i+1} \eta_i^{1-j}$ and $\beta_{i+1} \eta_i^{1-j}$ are conjugate, then they are conjugate under some $\delta_i \eta_i^a$, by Lemma 5.2. Since we picked only one element from each such conjugacy class, $\alpha_{i+1} = \beta_{i+1}$. But then $\alpha'_{i+1} = \beta'_{i+1}$ (as in case (b)), and $\chi = \chi^\#$. The number of choices for α is equal to the number of nonconjugate nonzero γ generating $k_{f_{i+1}}/k_{f_i}$ (the relation is that $(\alpha \eta_i^{1-j})^{e_{i+1}/e_i} = \gamma \eta_i^{(1-j)e_{i+1}/e_i}$). This number is $C_{j-1} = C_{t_{i+1}}$. This completes the proof of Theorem 6.1. □

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The next task is to compute which elements of G commute with χ on H_0. As before, we write $W_{(j)}$ for the group of power-permutation matrices in $G_{(j)}$, and W_i for those in G_i. Recall that $G_{(j)} = (K_{n/f_i} \cap G_{(j)}) W_{(j)} (K_{n/f_i} \cap G_{(j)})$, where i is the largest index with $t_i \geq j$. We have seen that $G_{(j)}$ commutes with $\chi|_{H_0^j}$ for $j > r_0$.

For each j, let $N^j = \{x \in Z_e K_e : \chi^x = \chi \text{ on } H_0^j\}$. By Theorem 6.1,

$$N^j = K_e^{c_1} (G_1 \cap K_e^{c_2}) \cdots (G_{i-1} \cap K_e^{c_i}) (G_{(j)} \cap Z_e K_e),$$

where i is the largest index with $t_i \geq j$ and $c_l = t_l''$ if $j \leq t_l'$, $c_l = t_l - j + 1$ otherwise.

(8.1) **Theorem.** *For all $j \geq r_0$, $\chi|_{H_0^j}$ intertwines with $\chi|_{x^{-1} H_0^j x}$ if and only if $x \in N^j G_{(j)} N^j$.*

Proof. We essentially proved "if" in Theorem 6.1, since N^j normalizes $H_0 \cap K_e^j$. The converse uses backwards induction on j. If $j = t_1$, the claim follows from Theorem 2.4 of [15]. For the inductive step, we assume the result for j and prove it for $j-1$. Let i be the largest index with $t_i \geq j$. Assume first that $t_{i+1} < j-1$. From the inductive hypothesis, $x \in N^j G_{(j)} N^j$. Write $G_{(j)} = (G_{(j)} \cap K_e) W_{(j)} (G_{(j)} \cap K_e)$ and use Lemmas 3.1–3.3 to see that $x \in N^j W_{(j-1)} N^j$. Write $x = k_1 g k_2^{-1}$, with $g \in W_{(j-1)}$ and k_1, $k_2 \in N^j$; set $\chi^{k_1} = \chi\chi_1$, $\chi^{k_2} = \chi\chi_2$, where χ_1, χ_2 are trivial on H_0^j and on $G_{(j)} \cap K_e^{j-1}$. (We saw in proving (11) in steps (4) and (5) of Theorem 6.1 that $\chi^{k_h}(y) = \chi(y)$ if $k_h \in N^j$ and $y \in G_{(j)} \cap K_e^{j-1}$.) For $y \in H_0^{j-1}$, choose $v \in H_0^{j-1}$ so that $k_2^{-1} v k_2 = y$; assume $gyg^{-1} \in H_0^{j-1}$. Then

$$\chi^{k_1}(gyg^{-1}) = \chi(k_1 g k_2^{-1} v k_2 g^{-1} k^{-1}) = \chi(v) = \chi^{k_2}(y),$$

or $\chi_1^g(y) = \chi_2(y)$. Let $H_{j-1} = (k_1 g)^{-1} (H_0^{j-1}) k_1 g \cap H_0^{j-1}$. After dividing k_1, k_l by terms known to commute with χ and to normalize H_{j-1}, we may write

$$k_l = (1 + \delta_{l,i-1} \eta_{i-1}^{t_i - j + 1}) k_l^*, \qquad l = 1, 2, \; k_l^* \in N^j \cap K_e^{t_i - j + 2}.$$

Assume that we show that $(1+\delta_{l,i-1}\eta^{t_i-j+1})g(1+\delta_{2,i-1}\eta^{t_i-j+1})^{-1}=k_1' g k_2'^{-1}$, where k_1', $k_2'\in N^{j-1}$. Again dividing by terms known to commute with χ and to normalize H_0^{j-1}, we get

$$k_l=(1+\delta_{l,i-2}\eta_{i-2}^{t_{i-1}-j+1})k_l^*,\qquad l=1,2,\ k_l^*\in N^j\cap K_e^{t_{i-1}-j+2},$$

and we need a similar argument. (The procedure stops at $1+\delta_{l,h-1}\eta_{h-1}^{t_h-j+1}$, where, as before, h is the smallest index with $t'_{h+1}\le j-1$.) So assume inductively that, modulo elements in N^j,

$$k_l=(1+\delta_{l,a-1}\eta_{a-1}^{t_a-j+1})k_l^*,\qquad l=1,2,\ k_l^*\in N^j\cap K_e^{t_a-j+2}.$$

Since g commutes with χ, we need to show that if $\chi_1(gyg^{-1})=\chi_2(y)$ for all $y\in H_0$, then $k_1gk_2^{-1}=k_1'gk_2'^{-1}$, $k_l'=(1+\delta'_{l,a-1}\eta_{a-1}^{t_a-j+1})k_l^{\#}$, $l=1,2$, with $k_l^{\#}\in N^j\cap K_e^{t_a-j+2}$ and $1+\delta'_{1,a-1}\eta_{a-1}^{t_a-j+1}$, $1+\delta'_{2,a-1}\eta_{a-1}^{t_a-j+1}\in G_a\cap K_e^{t_a})K_e^{t_{a+1}}$; this will extend the induction.

We want to apply Lemma 5.4. However, that lemma does not apply to the present situation, because χ is not of the fairly simple form assumed in the lemma. The idea of the proof is to create a new character, $\chi^{\#}$, on $G_a^{\sim}=G_{(t_a)}^{\sim}$ (the group of invertible elements of $M_{(t_a)}^{\sim}$; see Theorem 6.1, (8)(iv)), and to show that we can find a power-permutation matrix $g_0\in W_a^{\sim}$ (the corresponding group of power-permutation matrices) such that

$$(1+\delta_{1,a-1}\eta_{a-1}^{t_a-j+1})g_0(1+\delta_{2,a-1}\eta_{a-1}^{t_a-j+1})$$

commutes with $\chi^{\#}$. Lemma 5.4 will apply to this situation, and we will have

$$\begin{aligned} g_0&=k_0g, & k_0&\in N^{j-1}\quad\text{(by Lemma 3.3)},\\ k_1g_0k_2&=k_1'g_0k_2', & k_1',\,k_2'&\in G_a\cap K_e^{t_a-j+1}K_e^{t_a-j+2}. \end{aligned}$$

Then some simple algebra will complete the inductive step.

Here are the details. Let χ_0 be the extension of $\chi|_{H_0^{1+t_a}}$ to H_0^{j-1}, guaranteed by repeated use of Lemma 6.2, such that G_{a-1} commutes with χ_0. Restrict attention to $H_0^{t_a+1}(G_{a-1}\cap K_e^{j-1})=H^*$. On these elements, $\chi^{k_1}=\chi^{k_1^{\sim}}$, $k_1^{\sim}=1+\delta_{1,a-1}\eta_{a-1}^{t_a-j+1}$, and a similar claim holds for k_2; since k^*, $k^*\in N^j\cap K_e^{t_a-j+2}$ already, we may assume that $k_l=1+\delta_{l,a-1}\eta_{a-1}^{t_a-j+1}$, $l=1,2$. On the extension to $H^*\cap K_e^{t_a}$, $\chi=\chi_0\chi_*$, where $\chi_*(1+\gamma\eta_{a-1}^{t_a})=\psi\circ\mathrm{Tr}^{(e_i)}(\alpha\gamma^{\sigma^{-t_a}})$ and χ_* is trivial on $H^*\cap K_e^{t_a+1}$. Extend χ_* to χ_{*0} on H^* so as to commute with $G_a^{\sim}$ by using Lemma 6.2 repeatedly; let $\chi^{\#}=\chi_0\chi_{*0}$.

By using (5) and applying Lemma 3.3 to the subalgebra $M_{a-1}^{\sim}=M_{(t_a)}^{\sim}$ of M_{a-1} commuting with $\alpha\eta_{a-1}^{-t_a}$ and with F_{f_i} (we need the compositum with F_{f_i} to apply the lemma), we can find an element $g_0\in W_{a-1}^{\sim}$ such that $m=g^{-1}g_0\in K_e^1$. In fact, Lemmas 3.1 and 3.3 and Theorem 6.1(6) show that for $y\in H^*$, $mym^{-1}\in H_0$. Thus for any $y\in G_{a-1}\cap K_e^{j-1}$, $g_0^{-1}yg_0\in H^*\Leftrightarrow y\in H_0$. Furthermore, g_0 commutes with $\chi^{\#}$, and a calculation shows that $(\chi^{\#})^{k_1}=\chi^{\#}\chi_1$, $(\chi^{\#})^{k_2}=\chi^{\#}\chi_2$. Since χ_2 is defined on H_0^j and is trivial on H_0^{j-1}, and since $m\in K_e^1$, we have $\chi_2^m=\chi_2$; therefore $\chi_1^{g_0}=\chi_1^g=\chi_2$ on H^*, and so

$(\chi^{\#})^{k_1 g_0 k_2^{-1}} = \chi^{\#}$ where both are defined. Since G_{a-1} commutes with χ_0, we have $\chi_{*0}^{k_1 g_0 k_2^{-1}} = \chi_{*0}$ where both are defined.

Now Lemma 5.4 applies (with G_{a-1} playing the role of GL_n); we have $k_1 g_0 k_2^{-1} = k_1' g_0 k_2'^{-1}$, where $k_1', k_2' \in (K_e^{t_a-j} \cap G_a^{\sim})(G_{a-1} \cap K_e^{t_a-j+1}) \subseteq (K_e^{t_a-j} \cap G_a)(G_{a-1} \cap K_e^{t_a-j+1})$ (this last by Theorem 6.1(8) and Lemma 3.1). Use k to denote an element of $K_e^{t_a+1-j}$ (which may change from equation to equation). Then, since $(K_e^{t_a-j}, m) \subseteq K_e^{t_a-j+1}$ (and similarly for m^{-1}), we get

$$\begin{aligned} k_1 g k_2^{-1} &= k_1 g_0 k_2^{-1}(k_2 m^{-1} k_2^{-1} m) m^{-1} = k_1 g_0 k_2^{-1} k m^{-1} = k_1' g_0 k_2'^{-1} m^{-1} k \\ &= k_1' g_0 k_2'^{-1}(k_2' m^{-1} k_2'^{-1})(k_2', m) k = k_1' g k_2'^{-1} k, \end{aligned}$$

and the induction continues. This completes the proof when $j-1 \neq t_{i+1}$.

If $j-1 = t_{i+1}$, then restrict attention to $H_0^j(K_e^{j-1} \cap G_i)$; write $x = k_1 g k_2^{-1}$, with $g \in W_{(j)} = W_i$ and $k_1, k_2 \in N^j$. We have $\chi = \chi_0 \chi_1$, χ_0 as in Lemma 6.2 and $\chi_1(1 + \alpha\eta_i^j) = \psi \circ \mathrm{Tr}(\alpha\gamma^{\sigma^{1-j}})$, where $E_i[\alpha\eta_i^{1-j}]$ is a field of ramification index e_{i+1} and residue class degree f_{i+1} over F. Elements of N^j commute with χ_1, and $G_{(j)}$ commutes with χ_0; also, $\chi_0^{k_1} = \chi_0$ on $G_i \cap K_e^{j-1}$ (as we saw in the proof of (11) in Theorem 6.1, part (4)). So on the above elements, $\chi^x = \chi_0 \chi_1^g$, or $\chi_1^g = \chi_1$. Theorem 2.4 of [15] (or Lemma 6 of [10]) says that $g = k_1 g_1 k_2$, where $k_1, k_2 \in G_i \cap K_e^1$ and g_1 is a power-permutation matrix for the group of elements in G_i commuting with $E_{(j)}^{\sim}$. By Lemma 3.3, we may assume that $g_1 \in W_{(j-1)}$ at the cost of changing the k_i to elements of $(G_i \cap K_e^1) \cdot N^j \subseteq N^j$. The rest of the proof for this case is the same as in the previous one. $\square$

We now describe the basic building blocks for the supercuspidals. The following theorem does not give complete information on the set of elements commuting with the representations we construct, since (v) and (vi) apply only to the restriction to $H \cap K_e^1$.

(8.2) **Proposition.** *Given the (s_i, e_i, f_i) of (1.3), define the t_i, t_i', t_i'', and C_j as for Theorem 6.1. There are $e \cdot \prod_{j=0}^{t_1} C_j$ pairs (H, ρ), where H is a subgroup of $Z_e K_e$ and ρ is an irreducible representation of H, plus fields $E_{(j)}$, matrix subalgebras $M_{(j)}$, and groups $G_{(j)}$ = group of invertible elements of $M_{(j)}$, $0 \leq j \leq t_1$, such that (with $E_i = E_{(t_{i+1}+1)}$, $M_i = M_{(t_{i+1}+1)}$, $G_i = G_{(t_{i+1}+1)}$ for $1 \leq i < r$; $E_r = E_{t_r'}$, etc.):*

(i) *$e(E_i/F) = e_i$ and $f(E_i/F) = f_i$;*

(ii) *$M_{(j)}$ = algebra of elements commuting with $E_{(j)}$ (hence $M_r = E_r$);*

(iii) *$H = K_e^{t_1'}(K_e^{t_2'} \cap G_1) \cdots (K_e^{t_r'} \cap G_{r-1}) G_r$;*

(iv) *ρ is a character if $t_r > 0$, and a character on H tensored with a cuspidal representation of $K_e \cap G_{r-1}/K_e^1 \cap G_{r-1} \cong \mathrm{GL}_{f_{r-1}/f_r}(k_{f_{r-1}})$ (extended as a character to Z_e) if $t_r = 0$, and $K_e^{t_1+1}\langle\varpi_F\rangle \subseteq \mathrm{Ker}\,\rho$;*

(v) *for $j \geq 1$, set*

$$H^j = H \cap K^j, \quad N^j = K_{e^1}^{b_1}(K_{e^2}^{b_2} \cap G_1) \cdots (K_e^{b_i} \cap G_{i-1})(Z_e K_e \cap G_{(j)}),$$

where $b_l = \min(t_l'', t_l + 1 - j)$ and i = largest index with $t_i \geq j$. Then x commutes with $\rho|_{H^j}$ iff $x \in N^j W_{(j)} N^j$ (as before, $W_{(j)}$ = group of permutation-power matrices for $G_{(j)}$);

(vi) *distinct* (H, χ) *are nonconjugate in* Z_eK_e.

Proof. We construct the (H, χ) by a double induction on n and s_1. If $n = 1$, then there are no triples (s_i, e_i, f_i), and χ is trivial on $F^\times$. If $s_1 = 0$, then $r = 1$, $e_1 = 1$, and $f_1 = n$, and we take ρ to be a cuspidal representation of $K_1/K_1^1 \cong \mathrm{GL}_n(k)$, extended to be I on ϖ. It is known (see, e.g., [22]) that there are C_0 nonconjugate representations of this sort, all of dimension $\prod_{j=1}^{n-1}(q^j - 1)$. We set $E_1 = F_n$. The other parts are now immediate. (In fact, Theorem 4.1 of [3] shows that $\rho^x = \rho \Leftrightarrow x \in Z_1K_1$.)

Assume now that $s_1 \geq 1$, and let r_0 be the smallest index with $2s_{r_0+1} \leq s_{r_0}$. We use Theorem 6.1 to obtain pairs (H_0, χ) and fields E_i $(1 \leq i \leq r_0)$, together with the corresponding M_i, G_i; $H_0 = K_e^{t_1'}(K_e^{t_2'} \cap G_1) \cdots (K_e^{t_{r_0}'} \cap G_{r_0-1})$, and the set of elements commuting with χ is $J_0G_{r_0}J_0$, where

$$J_0 = K_e^{t_1''}(K_e^{t_2''} \cap G_1) \cdots (K_e^{t_{r_0}''} \cap G_{r_0-1}).$$

If $r_0 = r$, then $G_{r_0} = E_r^\times$ and $E_r^\times$ normalizes J_0. Let $E_{(j)} = E_r$ for $j < r$, and extend χ to a character ρ on $H = H_0E_r^\times$ trivial on ϖ; this is possible in $[H : H_0\langle\varpi\rangle] = e \cdot \prod_{j=0}^{t_{r_0}-1} C_i$ ways. Because $E_r^\times$ commutes with χ and normalizes J_0, one checks easily that $E_r^\times J_0$ commutes with ρ; indeed, $\rho^x = \rho \Leftrightarrow x \in E_r^\times N^1$. This completes the proof in this case.

If $r_0 < r$, we extend χ to a character on $H_0(G_{r_0} \cap Z_eK_e)$ that is trivial on $(G_{r_0}, G_{r_0}) \cap Z_eK_e$ and is hence fixed by G_{r_0}; note that $(G_{r_0} \cap Z_eK_e)$ normalizes H_0, by Theorem 6.1. Lemma 4.1 says that the extension is possible, because any element of $(G_{r_0}, G_{r_0}) \cap H_0$ is a product of commutators in $(G_{r_0} \cap Z_eK_e, G_{r_0} \cap H_0)$, and we know that χ is 1 on these elements. There are $\prod_{j=t_{r_0+1}+1}^{t_{r_0}-1} C_j$ different restrictions of these extensions to $H_0(G_{r_0} \cap K_e^{t_{r_0+1}+1})$, as one sees by calculating the index of the commutator subgroup. (If χ_1 is one such character defined on $H_0(G_{r_0} \cap K_e^j)$, the number of such extensions of χ_1 to $H_0(G_{r_0} \cap K_e^{j-1})$ is $[G_{r_0} \cap K_e^{j-1} : (G_{r_0} \cap K_e^j)(\mathrm{SL}_{r_0} \cap K_e^{j-1})]$, where SL_{r_0} is the special linear group corresponding to G_{r_0}; this index is $q^{f_{r_0}}$ if $(j-1)e_{r_0}$ is a multiple of e and 1 otherwise.) Choose one extension (to be called χ) for each restriction, and form the tensor product $\rho = \chi \otimes \rho_0$, where ρ_0 is a representation of a subgroup H_1 of $G_{r_0} \cap Z_eK_e$ satisfying (i)–(vi) for the triples $(s_i/f_{r_0}, e_i/e_{r_0}, f_i/f_{r_0})$, $r_0 + 1 \leq i \leq r$, constructed via the inductive hypothesis (and extended to be trivial on $H_0(G_{r_0} \cap K_e^{t_{r_0+1}+1})$; $H = H_0H_1$, and the $E_{(j)}$, $M_{(j)}$, $G_{(j)}$ are as defined inductively for ρ_0. One checks that there are $e \cdot \prod_{j=0}^{t_{r_0+1}} C_j$ such representations (note that these representations need be trivial only on ϖ, not on $\eta_{r_0}^{n/e_{r_0}f_{r_0}}$).

The points that still need checking are (v) and (vi). Let $H^1 = H \cap K_e^1$ and $\rho^1 = \rho|_{H_1}$. Then $J_0 = K_e^{t_1''}(K_e^{t_2''} \cap G_1) \cdots (K_e^{t_{r_0}''} \cap G_{r_0-1})$ commutes with ρ. For if $y \in H_0(G_{r_0} \cap K_e^1)$ and $w \in K_e^{t_i''} \cap G_{i-1}$, then $\chi(ywy^{-1}w^{-1}) = 1$ by Lemma 4.2(b) because $2t_i'' + 1 \geq t_i + 1$ and w commutes with χ on $H_0^{t_i+1}$. So to prove (v), we need (in view of Theorem 6.1(7)) only determine which elements of G_{r_0} commute with ρ^1, and now (v) follows from the induction hypothesis. The argument for (vi) is essentially the same. If ρ^1, $\rho^{\sim 1}$ are conjugate, then they agree on H_0, by Theorem 6.1, and, since $N^{t_{r_0}'} = N^{t_{r_0+1}+1}$, they agree on $H \cap K_e^{t_{r_0+1}+1}$. On this group, both are multiples of the same character χ. We have fixed an extension of χ to H^1, which we also call χ, and $\rho \otimes \chi^{-1}$, $\rho^\sim \otimes \chi^{-1}$ are conjugate. Now the inductive hypothesis proves that $\rho^1 \cong \rho^{\sim 1}$. □

Remark. We have not shown that representations corresponding to different sequences of triples are nonconjugate. For fixed e and f, different sequences of s_i lead to pairs (H, ρ) whose restrictions to their respective H^1 are nonconjugate over $Z_e K_e$ because one can read off the s_i from the indices $[H \cap K_e^j : H \cap K_e^{j+1}]$; since the s_i determine the e_i, we can have conjugacy only if the sequences of s_i and e_i agree. For the f_i, one can use the arguments proving (11) in the proof of Theorem 6.1 or use the fact that the indices $[H \cap K_e^j : H \cap K_e^{j+1}]$, $[N^j : N^{j+1}]$ determine the f_i. We say more about this in the next section, and we deal with different e, f in §10.

9

The next step is to extend ρ to a representation ρ_1 on J that is a multiple of ρ on $H \cap K_e^1$. The procedure is fairly standard and goes back to [10]; here is a sketch. Suppose for definiteness that t_1 is even (if it is odd, then $t_1' = t_1''$ and there is nothing to do at this step). Pick a cross-section for $HK_e^{t_1''}/H$, and extend ρ to $HK_e^{t_1''}$ by making it I on the cross-section. Then ρ is a projective representation on $HK_e^{t_1''}$. The cocycle defining the multiplier projects to a cocycle on $HK_e^{t_1''}/H \cap K_e^1$. This group can be regarded as a subgroup of the semidirect product of a symplectic group with $K_e^{t_1''}/H \cap K_e^{t_1''}$. When p is odd, the cocycle is the inverse of that for the Weil (or oscillator) representation; a unified treatment (including $p = 2$) is given in [8], and the cocycle is computed explicitly in [5] for the case of division algebras. Tensor ρ with the Weil representation λ_1 to produce an ordinary irreducible representation on $HK_e^{t_1''}$. Repeat this construction for all even $t_i > 0$ to get ρ_1. Further details are given in the Appendix to this paper. As noted there, distinct ρ produce distinct ρ_1.

(9.1) **Theorem.** (a) *Let ρ_1 be as above. Then $\pi = \mathrm{Ind}_J^G \rho_1$ is irreducible.*

(b) *Let (ρ_1, J) and $(\rho_1^\sim, J^\sim)$ be distinct representations, constructed as above on subgroups of $Z_e K_e$ from completely satisfactory representations. Then the induced representations π and $\pi^\sim$ on G are distinct.*

Proof. (a) From [17], we need to show that if $x \in G$ is such that ρ_1^x and ρ_1 intertwine on their common domain, then $x \in J$; Proposition 8.2 says that if $s_r > 0$, this is already true for $\rho_1|_{H^1}$. When $s_r = 0$, Proposition 8.2 says that $x \in J^1 G_{r-1} J^1$. Since $J^1 \subseteq J$, we need only consider $x \in G_{r-1}$. The argument of Lemma 14 of [10] shows that x also intertwines ρ_0 with itself, where ρ_0 is the lift of the cuspidal representation of $\mathrm{GL}_{n_{r-1}}(k_{f_{r-1}})$ used in constructing ρ. From Theorem 4.1 of [3], $x \in J$.

For (b), let the triples associated with $\rho_1^\sim$ be $(s_i^\sim, e_i^\sim, f_i^\sim)$, $1 \le i \le r^\sim$; we assume that $e_{r^\sim} = e$. Suppose that ρ_1^x and $\rho_1^\sim$ intertwine on their common domain. By restricting to $H \cap K_e^1$ and $H^\sim \cap K_e^1$, we may assume that ρ^x and $\rho^\sim$ intertwine on their common domain. Without loss of generality, we may assume that $s_1 \ge s_1^\sim$. Assume that $\rho(1+y) = \psi \circ \mathrm{Tr}(\alpha \varpi^{-t_1} y)$ and $\rho^\sim(1+y) = \psi \circ$

$\operatorname{Tr}(\alpha^{\sim}\varpi^{-t_1}y)$ for $y \in A_e^{s_1}$. From Proposition 5.3, $(s_1, e_1, f_1) = (s_1^{\sim}, e_1^{\sim}, f_1^{\sim})$ and $\alpha\varpi^{-t_1}$, $\alpha^{\sim}\varpi^{-t_1}$ are conjugate by an element of Z_eK_e. Since we chose only one element from each conjugacy class, $\rho = \rho^{\sim}$ on $K_e^{t_1} = K_e^{t_1} \cap H = K_e^{t_1} \cap H^{\sim}$.

Assume inductively that $(s_l, e_l, f_l) = (s_l^{\sim}, e_l^{\sim}, f_l^{\sim})$ for $l \leq i$ (where i is the largest index with $s_i \geq j$), that $E_{(h)} = E_{(h)}^{\sim}$ for $h \leq j$ (so that, in particular, $K_e^j \cap J = K_e^j \cap J^{\sim}$), and that $\rho = \rho^{\sim}$ on this group. We want to extend this result to $j-1$. From Proposition 8.2, $x \in N^j W_{(j)} N^j$. Let $N_1^j = N^j \cap K_{n/f_i}^1$. Then $W_{(j)}$ contains a set of coset representatives for N^j/N_1^j, so that $x \in N_1^j W_{(j)} N_1^j$. Write $x = kbk'$ $(k, k' \in N_1^j,\ b \in W_{(j)})$, let $\rho|_{K_e^{j-1} \cap H}$ be a multiple of χ, and write $\chi = \chi_0\chi_{\alpha'} = \chi_0\chi'$ as in the proof of Theorem 6.1; write $\chi^{\sim} = \chi_0\chi_{\beta'}$ correspondingly.

We saw in the proof of Theorem 6.1 (especially in the construction of $\eta_{(j-1)}$) that for $y \in K_e^{j-1} \cap G_{(j)}$ and $k_0 \in N_1^j$, $\chi(k_0 y k_0^{-1}) = \chi(y)$; similarly, $\chi^{\sim}(k_0 y k_0^{-1}) = \chi^{\sim}(y)$. For $w \in k^{-1}(K_e^{j-1} \cap G_{(j)})k$, let $y = kwk^{-1}$; then

$$\chi^{\sim k'^{-1}}(w) = \chi^{\sim}(k'^{-1}k^{-1}ykk') = \chi^{\sim}(y),$$

by the above result with $k_0 = (kk')^{-1}$. Since $(\chi^k)^b = (\chi^{\sim})^{k'-1}$ on their common domain and $\chi^k(w) = \chi(y)$, we get

$$\chi^{\sim}(y) = \chi^{\sim k'^{-1}}(w) = \chi^{kb}(w) = \chi^b(y), \qquad \text{all } y \in K_e^{j-1} \cap G_{(j)} \cap b^{-1}(K_e^{j-1} \cap G_{(j)})b.$$

Since $G_{(j)}$ commutes with χ_0, we have $\chi_0^b = \chi_0$. Therefore $\chi_{\alpha'}^b = \chi_{\beta'}$ on these elements. But for $y \in K_e^{j-1} \cap G_{(j)}$,

$$\chi_{\alpha'}(y) = \psi \circ \operatorname{Tr}^{(e_i)}(y\alpha^{\sigma^{-j}}), \qquad \chi_{\beta'}(y) = \psi \circ \operatorname{Tr}^{(e_i)}(y\beta^{\sigma^{-j}}),$$

where $\alpha = \operatorname{Tr}_{e_i}\alpha'$ and $\beta = \operatorname{Tr}_{e_i}\beta'$. Since $\alpha\eta_{(j)}^{1-j}$ and $\beta\eta_{(j)}^{1-j}$ generate fields over $E_{(j)}$, Proposition 5.3 implies that $\alpha\eta_{(j)}^{1-j}$ and $\beta\eta_{(j)}^{1-j}$ are conjugate. Because we picked only one element from each conjugacy class, $\alpha = \beta$; because we picked only one α' for each α, $\chi_{\alpha'} = \chi_{\beta'}$. This also shows that $t_{i+1} = j-1$ iff $t_{i+1}^{\sim} = j-1$ and that $e_{i+1} = e_{i+1}^{\sim}$, $f_{i+1} = f_{i+1}^{\sim}$ in that case. This extends the induction.

When $s_r = 0$, this forces $s_r^{\sim} = 0$. As in (a) we show that ρ_1^x intertwines $\rho_1^{\sim}$ only if x intertwines the corresponding cuspidal representations of ρ_0, $\rho_0^{\sim}$ by invoking Lemma 14 of [10]; then Theorem 4.1 of [3] shows that we must have $\rho_0 \cong \rho_0^{\sim}$ and hence $\rho \cong \rho^{\sim}$. This completes the proof. □

10

This section is devoted to computing the formal degrees of the supercuspidal representations computed in the previous section. This is done in three parts. We begin with the data $(s_1, e_1, f_1), \ldots, (s_r, e_r, f_r)$, and compute $[Z_eK_e : J]$, J as defined in (4.5). Then we compute the degree of the irreducible representation $\tau = \operatorname{Ind}_{J \to Z_eK_e} \rho_1$, ρ_1 as defined just before Theorem 8.2. Finally, we compute the formal degree of the supercuspidal π induced from τ. The calculations are similar to those in §2 of [6].

(10.1) **Lemma.** *Use notation as in §§6–8. Then*

$$[Z_eK_e : J] = \begin{cases} q^{a_0}c(f)^e/c(f/f_{r-1}) & \text{if } s_r = 0, \\ q^{a_0}c(f)^e/(q^f-1) & \text{if } s_r > 0, \end{cases}$$

where $c(f/f_i) = (q^f-1)(q^{f_i(f/f_i-1)}-1)(q^{f_i(f/f_i-2)}-1)\cdots(q^{f_i}-1)$, $c(f) = c(f/1)$,

$$a_0 = \begin{cases} \sum_{i=1}^{r} ft_i''(n_{i-1}-n_i) + n(f-1)/2 - nf + f & \text{if } s_r > 0, \\ \sum_{i=1}^{r} ft_i''(n_{i-1}-n_i) + n(f-1)/2 - f(f/f_{r-1}-1)/2 - fn + fn_{r-1} & \\ & \text{if } s_r = 0; \end{cases}$$

$n_i = n/e_if_i$, *and* $e_0 = f_0 = 1$.

Proof. We have

$$[Z_eK_e : J] = [Z_eK_e : JK_e][JK_e : J],$$
$$[JK_e : J] = [K_e : J\cap K_e] = [K_e : (J\cap K_e)K_e^1][(J\cap K_e)K_e^1 : J\cap K_e],$$
$$[(J\cap K_e)K_e^1 : J\cap K_e] = [K_e^1 : J\cap K_e^1],$$

and so on; hence

$$[Z_eK_e : J] = [Z_eK_e : JK_e][K_e : (J\cap K_e)K_e^1]\prod_{j=1}^{\infty}[K_e^j : (J\cap K_e^j)K_e^{j+1}].$$

(The terms in the product are 1 for $j \geq t_1''$.) Since $Z_eK_e \cap G_r$ contains a generator of A_e^1 (regarded as an ideal of A_e), we have

$$[Z_eK_e : JK_e] = 1.$$

Next,

$$[K_e : (J\cap K_e)K_e^1] = [m_e^\times : m_e^\times \cap J];$$

$m_e^\times$ has $q^{ef(f-1)/2}c(f)^e$ elements. If $s_r > 0$, $m_e^\times \cap J = (k_f)^\times$ has $q^f - 1$ elements; if $s_r = 0$, $m_e^\times \cap J = \mathrm{GL}_{f/f_{r-1}}(k_{f_{r-1}})$ has $c(f/f_{r-1})q^{f(f/f_{r-1}-1)/2}$ elements. Finally, if $t_{i+1}'' \leq j < t_i''$ (we take $t_{r+1}'' = 0$), then

$$[K_e^j : (J_0\cap K_e^j)K_e^{j+1}] = [m_e : m_e \cap M_i] = q^{ef^2(1-1/e_if_i)}.$$

Hence if $s_r > 0$, then

$$[Z_eK_e : J] = q^{a_0}c(f)^e/(q^f-1),$$

where

$$a_0 = ef(f-1)/2 + ef^2(t_1''-1) - \sum_{i=1}^{r} ef^2(t_i''-t_{i+1}'')/e_if_i + ef^2/e_rf_r;$$

if $s_r = 0$, then

$$[Z_eK_e : J] = q^{a_0}c(f)^e/c(f/f_{r-1}),$$

where (since $e_r = e_{r-1}$ in this case)

$$a_0 = ef(f-1)/2 + ef^2(t_1''-1) - f(f/f_{r-1}-1)/2$$
$$- \sum_{i=1}^{r-1} ef^2(t_i''-t_{i+1}'')/e_if_i + ef^2/e_rf_{r-1}.$$

In this last expression, we can sum from 1 to r because $t_r'' = t_{r+1}'' = 0$. Now use partial summation and note that $e_r f_r = ef = n$. $\square$

We now consider $\dim \rho_1$, ρ_1 as defined before Theorem 8.2. First of all, ρ (as in Lemma 8.1) has dimension 1 if $s_r > 0$ and dimension $q(f/f_{r-1})/(q^f - 1)$ if $s_r = 0$ (see, e.g., [22]). Second, we tensor by a representation of dimension $[m_e^{f_{i-1}}(e_{i-1}) : m_e^{f_i}(e_i)]^{1/2} = q^{f(n_{i-1}-n_i)/2}$ if t_i is even and > 0 and by a representation of dimension 1 if t_i is odd or 0. Combining this with Lemma 10.1, we get

$$\text{Dim } \tau = \frac{c(f)^e}{q^f - 1} q^{a_1}, \tag{10.2}$$

where, for $s_r > 0$,

$$\begin{aligned} a_1 &= \sum_{i=1}^{r} f t_i''(n_{i-1} - n_i) + \frac{n(f-1)}{2} - nf + f + \sum_{t_i \text{ even}} f(n_{i-1} - n_i)/2 \\ &= \sum_{i=1}^{r} f\left(\frac{t_i + 1}{2}\right)(n_{i-1} - n_i) - \frac{n(f+1)}{2} + f \end{aligned} \tag{10.3}$$

(since $t_i'' = (t_i + 1)/2$ if t_i is odd and $t_i'' = t_i/2$ if $t_i \neq 0$ is even)

$$= \sum_{i=1}^{r} f \frac{t_i}{2}(n_{i-1} - n_i) - \frac{n-f}{2}; \tag{10.4}$$

for $s_r = 0$, we have (similarly)

$$\begin{aligned} a_1 &= \sum_{i=1}^{r} f t_i''(n_{i-1} - n_i) - \frac{n(f-1)}{2} - \frac{f(f/f_{r-1} - 1)}{2} \\ &\quad - fn + fn_{r-1} + \sum_{\substack{t_i \text{ even} \\ \neq 0}} \frac{f(n_{i-1} - n_i)}{2} \\ &= \sum_{i=1}^{r} f\left(\frac{t_i + 1}{2}\right)(n_{i-1} - n_i) - \frac{fn}{2} - \frac{(n-f)}{2} + f \end{aligned}$$

(since $n_{r-1} = f/f_{r-1}$)

$$= \sum_{i=1}^{r} f \frac{t_i}{2}(n_{i-1} - n_i) - \frac{n-f}{2},$$

just as in (10.4). Thus we have proved:

(10.5) **Lemma.** *Use notation as in* §§6–9 *and as above. For the representation* $\tau = \text{Ind}_{J_1 \to Z_e K_e} \rho_1$,

$$\text{Dim } \tau = \frac{c(f)^e}{q^f - 1} q^{a_1}, \qquad a_1 \textit{ as in } (10.4),$$

where $e_i f_i n_i = n$. $\square$

We can now compute the formal degree of the supercuspidal π induced from τ. The appropriate normalization of Haar measure for G (or, more properly, G/Z_1) is that which gives the Steinberg representation formal degree 1. It is standard (see, e.g., [3] or [21]) that this measure gives K_1Z_1/Z_1 volume $n^{-1}(q^n-1)^{-1}c(n) = n^{-1}\prod_{j=1}^{n-1}(q^j-1)$. Lemma 2.2.7 of [6] then shows that

$$\operatorname{vol}(Z_eK_e/Z_1) = ec(f)^e c(n)^{-1}\operatorname{vol}(Z_1K_1/Z_1) = f^{-1}(q^n-1)^{-1}c(f)^e.$$

Combining this with (10.4), we get

(10.6) **Theorem.** *Any supercuspidal representation* π *associated with data* (s_i, e_i, f_i), $1 \le i \le r$, *has formal degree*

$$f\frac{q^n-1}{q^f-1}q^{a_1},$$

a_1 *as in* (10.4), *where* $t_i = s_i/f$. □

Remark. This formula is the same as that of Theorem 2.28 of [6]; in that paper, $j_i = (t_i+1)/2$. It is easiest to use (10.3) to compare the results.

We need one important corollary of Theorem 10.6.

(10.7) **Theorem.** *Let* π *be a supercuspidal representation of* $\mathrm{GL}_n(F)$ *associated with the data* (s_i, e_i, f_i), $1 \le i \le r$; *let* $\pi^\sim$ *be another supercuspidal, associated with the data* $(s_i^\sim, e_i^\sim, f_i^\sim)$, $1 \le i \le r^\sim$. *Let* $f = f_r$ *and* $f^\sim = f^\sim_{r^\sim}$. *Suppose that* $f \ne f^\sim$. *Then* π *and* $\pi^\sim$ *are inequivalent.*

Proof. Let $f = f_0 f_*$, where $(f_0, p) = 1$ and f_* is a power of p; write $f^\sim = f_0^\sim f_*^\sim$ similarly. We may assume that $f_* | f_*^\sim$. Then the largest factors in the formal degree of π, $\pi^\sim$ prime to p are respectively

$$f_0\frac{q^n-1}{q^f-1}, \qquad f_0^\sim\frac{q^n-1}{q^{f^\sim}-1}.$$

By Lemma 4.1 of [6], these numbers are unequal if $f_0 \ne f_0^\sim$. □

11

In this section we complete the proof of Theorem 1.1 by proving:

(11.1) **Theorem.** *Every supercuspidal representation of* $\mathrm{GL}_n(F)$ *is obtained by tensoring a character with one of those constructed in Theorem* 8.2.

Proof. Since every supercuspidal is the tensor product of a unitary supercuspidal trivial on ϖ_F with a quasicharacter, we restrict attention in what follows to this class of supercuspidals. Let $\mathscr{E}^0(G)$ and $\mathscr{E}^{\mathrm{sp}}(G)$ denote respectively the sets of (equivalence classes of) supercuspidal representations and of (equivalence classes of) discrete series, but not supercuspidal (= generalized special) representations of G trivial on ϖ_F. (In the rest of this proof, we do not distinguish between a representation and its equivalence class.) For an integer $m > 0$, write $\mathscr{E}_m^0(G)$ and $\mathscr{E}_m^{\mathrm{sp}}(G)$ for the subsets of $\mathscr{E}^0(G)$ and $\mathscr{E}^{\mathrm{sp}}(G)$ respectively with conductoral exponent $\le mn$. These, as we shall see, have finite cardinality.

We have already constructed certain elements of $\mathscr{E}_m^0(G)$. For a sequence $S = \{(s_1, e_1, f_1), \dots, (s_r, e_r, f_r)\}$ of triples, define $\alpha(S) = s_1$ and $n(S) = e_rf_r = n$. We have associated to S a sequence C_S of supercuspidal representations; its cardinality, $|C_S|$, is given in Theorem 8.2. It is proved in [2] that if $\pi \in C_S$, then the conductoral exponent $c(\pi)$ of π is $s_1 = \alpha(s)$. We construct other supercuspidals by tensoring these π with characters. To avoid repetitions, we proceed as follows: $(G \cap K_n^j)/(G \cap K_n^{j+1})(\mathrm{SL}_n(F) \cap K_n^j)$ has one element if $n \nmid j$ and q elements if $n|j$. In the latter case we write $j = mn$ and choose q characters $\chi_{m,1}, \dots, \chi_{m,q}$ of G, trivial on $G \cap K_n^{mn+1}$ and distinct on $G \cap K_n^{mn}$; we take $\chi_{m,1}$ to be trivial. If $mn > s_1$, we let a be the smallest multiple of n that is $> s_1$ and let $C_S(m) = \{\tau = \pi \otimes \chi_{a,j_a} \otimes \cdots \otimes \chi_{m,j_m} : 1 \leq j_h \leq q, \pi \in C_S\}$; for convenience, we write $C_S(m) = C_S$ if $mn = \alpha(S)$. Then if $T = \{(s_1^*, e_1^*, f_1^*), \dots, (s_{r^*}^*, e_{r^*}^*, f_{r^*}^*)\}$, we have $C_S(m) \cap C_T(m) \neq \varnothing$ only if $S = T$, and the elements of $C_S(m)$ constructed above are all distinct. Proof: if $e_r \neq e_{r^*}^*$, the formal degrees differ, so we may suppose that $e_r = e_{r^*}^* = e$. Suppose that $\tau_1 = \pi_1 \otimes \chi_{a,j_a} \otimes \cdots \otimes \chi_{m,j_m} \in C_S(m)$ and that $\tau_2 = \pi_2 \otimes \chi_{b,h_b} \otimes \cdots \otimes \chi_{m,h_m} \in C_T(m)$. Then τ_1 is induced from $\rho_1^\sim = \rho_1 \otimes \chi_{a,j_a} \otimes \cdots \otimes \chi_{m,j_m}$, where we also write χ_{a,j_a} for its restriction to the group J_1 on which ρ_1 is defined; similarly, τ_2 is induced from $\rho_2^\sim$. If $s_1, s_1^* = mn$, then there are no characters χ_{c,h_c} in the tensor product and we already know the result. If $s_1^* = mn > s_1$, then $\rho_1^\sim$ is a multiple of χ_{m,j_m} on K_e^{me}, while $\rho_2^\sim$ is a multiple of ρ_2 there; Lemma 5.3 implies that $\rho_1^\sim$, $\rho_2^\sim$ are not conjugate, so that τ_1 and τ_2 are not conjugate. If $s_1, s_1^* < mn$, then $\rho_1^\sim$ is a multiple of χ_{m,j_m} on K_e^{me}, while $\rho_2^\sim$ is a multiple of χ_{m,h_m} there; Lemma 5.3 implies that $\chi_{m,j_m} = \chi_{m,h_m}$ on $K_e^{me} \supseteq K_n^{mn}$. Therefore $j_m = h_m$. Tensor with χ_{m,j_m}^{-1} and continue inductively.

If $\tau \neq \pi$, the conductor $c(\tau)$ of τ is shown in [2] to be $m_0 n$, where m_0 is the largest index with $h_{m_0} \neq 1$. (If $\tau = \pi$, we have already computed $c(\tau)$.) Thus all members of $C_S(m)$ lie in $\mathscr{E}_m^0(G)$. Let $\mathscr{E}_m^1(G) = \{\tau \in G^\wedge : \tau \in C_S(m)$ for some S with $\alpha(S) \leq m\}$. Since $|C_S(m)| = |C_S| q^{m-[\alpha(S)/n]}$, we see that

$$|\mathscr{E}_m^1(G)| = \sum_{\alpha(S) \leq m} |C_S| q^{m-[\alpha(S)/n]}.$$

We want to prove that $|\mathscr{E}_m^1(G)| = |\mathscr{E}_m^0(G)|$. We argue by induction on n, the case $n = 1$ being trivial.

Let $D = D_n$ be a central division algebra of dimension n^2 over its center F. The unitary dual of the multiplicative group $D^\times$ was constructed in [4]. The representations in $(D^\times)^\wedge$ are classified by sequences $S = \{(s_1, e_1, f_1), \dots, (s_r, e_r f_r)\}$ satisfying (1.3), except that we require only the $e_r f_r | n$. Let $n(S) = e_r f_r$ (this is consistent with the previous definition of $n(S)$). Corresponding to S there is a collection of representations $C'_S \subseteq (D^\times)^\wedge$; the elements $\pi' \in C'_S$ all have conductors $c(\pi') = s_1 = \alpha(S)$. There are other elements of $(D^\times)^\wedge$ (again, we consider only representations trivial on ϖ_F), formed by tensoring with characters $\chi'_{a,h_a}, \dots$, just as for G. If we define $C'_S(m)$ in a manner analogous to $C_S(m)$ (i.e., one tensor with $\chi'_{a,h_a}, \dots, \chi'_{m,h_m}$), then the formula for $c(\tau')$, $\tau' \in C'_S(m)$, is the same as that for elements of $C_S(m)$. In particular, elements of $C'_S(m)$ have conductors $\leq mn$, and elements of $C'_S(m_1)$, $m_1 > m$, have conductors $\leq mn$ iff they are in $C'_S(m)$. Furthermore, $|C'_S(m)| = |C_S(m)|$ when $n(S) = n$; this follows from Theorem 5.5 of [4]. Thus if we let $\mathscr{F}_m^n(D_n) = \{\tau \in (D_n^\times)^\wedge : \tau \in C'_S(m)$ for some S with $n(S) = n\}$, then $|\mathscr{F}_n{}^m(D_n)| = |\mathscr{E}_m^1(G)|$.

Let $\mathcal{F}_m^{n_0}(D^n) = \{\tau \in (D_n^\times)^\wedge : \tau \in C'_S(m)$ for some S with $n(S) = n_0\}$, where $n_0|n$. (For $n_0 = 1$, $S = \varnothing$, and $C_\varnothing$ is the set of characters trivial on the first congruence subgroup K^1 that factor through the reduced norm map. There are $q-1$ such characters; thus $|C_\varnothing| = q-1$ and $|C_\varnothing(m)| = (q-1)q^m$.) One can see directly that $|\mathcal{F}_m^{n_0}(D_n)| = |\mathcal{F}_m^{n_0}(D_{n_0})|$; if $n_0|n$, $|\mathcal{F}_m^{n_0}(D_{n_0})| = |\mathcal{E}_m^0(\mathrm{GL}_{n_0}(F))|$, by the inductive hypothesis. Let $\mathcal{E}_m^{\mathrm{sp}}(G)$ be the set of irreducible square-integrable, nonsupercuspidal representations of G with conductor $\leq mn$ (i.e., the set of elements of $\mathcal{E}^{\mathrm{sp}}(G)$ with conductor $\leq mn$), then

$$|\mathcal{E}_m^{\mathrm{sp}}(G)| = \sum_{n_0|n\,,\, n_0 \neq n} |\mathcal{E}_m^0(\mathrm{GL}_{n_0}(F)|,$$

by [23] (plus Theorem 3.4 of [9], for calculation of conductors). Since

$$|\mathcal{E}_m^{\mathrm{sp}}(G)| + |\mathcal{E}_m^0(G)| = \sum_{n_0|n} |\mathcal{F}_m^{n_0}(D_n)| = |\mathcal{E}_m^{\mathrm{sp}}(G)| + |\mathcal{E}_m^1(G)|$$

by what we have just proved, $|\mathcal{E}_m^0(G)| = |\mathcal{E}_m^1(G)|$ and the induction extends. This proves the theorem when $\mathrm{char}\, F = 0$.

For the case $\mathrm{char}\, F = p$, we use [13]. Let $m > 0$. Then there is a field F_0, $\mathrm{char}\, F_0 = 0$, such that the Hecke algebras $\mathcal{H}(\mathrm{GL}_n(F_0)//K_1^{mn+1})$, $\mathcal{H}(\mathrm{GL}_n(F)//K_1^{mn+1})$ are isomorphic. (If $F = \mathbb{F}_q((x))$, $q = p^b$, then let $F_0 = \mathbb{Q}_p(\alpha, \varpi_0)$, where $\mathbb{Q}_p(\alpha)$ is unramified of degree b and $\varpi_0^N = p$ for some large N.) From [13], there is a 1-1 correspondence between the discrete series representations of $\mathrm{GL}_n(F_0)$ with a K_1^{mn+1}-fixed vector and those of $\mathrm{GL}_n(F)$ with a K_1^{mn+1}-fixed vector. (The Hecke algebra isomorphism constructed in [13] is obviously an isometry for appropriate choices of Haar measure. As noted in [12], this gives the correspondence for discrete series representations.) A discrete series representation π with conductor $\leq mn$ has a K_1^{mn+1}-fixed vector, since it is induced from a representation on an open subgroup with such a vector. Conversely, if any of the representations we have constructed have K_1^{mn+1}-fixed vectors, they have conductor $\leq mn$. For, in the terminology of [11], if the representation π has conductor j, then π has a minimal K-type of level j/n. The uniqueness property of minimal K-types says that π has no K_1^j-fixed vectors. (See also [2].) Since the discrete representations we have constructed with conductors $\leq mn$ for the two groups correspond 1-1, the theorem for $\mathrm{GL}_n(F)$ follows. □

Remarks. 1. In [14], Koch computed $\sum_{n_0|n} |\mathcal{F}_m^{n_0}(D)|$ by calculating the number of conjugacy classes in $D^\times/\langle\varpi\rangle K^{m+1}$, where K^{m+1} is the $(m+1)$th congruence subgroup. One can group conjugacy classes by sequences S with properties like (1.3), except that $e_r f_r | n$ and the s_i increase. Roughly speaking, for the conjugacy class represented by x there are elements x_i such that $F[x_i]/F$ has ramification index e_i and residue class degree f_i and such that $x_i^{-1}x \in P^{s_i}$ (where P is the prime ideal for the integers of D_i), but no such congruence (with e_i, f_i) is possible mod P^{s_i+1}. It is not hard to use results in [14] to show that $|C'_S(m)|$ is the number of conjugacy classes corresponding to sequences $(mn - s_1, e_1, f_1), \ldots, (mn - s_r, e_r, f_r)$. This would give a different way of completing the counting argument given above.

2. It is likely that the results of [12] on Hecke algebra isomorphisms apply also to the wildly ramified case. If they did, one could also prove Theorem

11.1 by comparing formal degrees in $(D_n^\times)_s^\frown$ and in $\mathscr{E}_s^0(\mathrm{GL}_n) \cup \mathscr{E}_s^{\mathrm{sp}}(\mathrm{GL}_n)$. In fact, the formal degrees of the supercuspidals in Theorem 8.2 are the same as the formal degrees of the irreducible representations of $D_n^\times$ wit the same data $(s_1, e_1, f_1), \ldots, (s_r, e_r, f_r)$, constructed in [4]. This means that once the Hecke algebra isomorphism theorem is extended, one has the following extension of the results in [6]: for any n and s_0, there is an h_0 such that for $q = p^h$ with $h \geq h_0$, the correspondence of the Matching Theorem associates representations of $D_n^\times$ and $\mathrm{GL}_n(F)$ with the same data $(s_1, e_1, f_1), \ldots, (s_r, e_r, f_r)$ whenever $s_1 \leq s_0$.

Appendix

Here is the cocycle calculation mentioned at the start of §9.

We assume inductively that we have extended ρ to a representation (which we also call ρ) on $H(J \cap K_e^{t_i'})$ that is a multiple of a character χ on $H \cap K_e^1$. We want to extend this situation to $H(J \cap K_e^{t_i''}) = H(J \cap K_e^{t_{i+1}'})$. Write $H_i = H(J \cap K_e^{t_i'})$, $J_i = H(J \cap K_e^{t_i''})$, $H_i^j = H_i \cap K_e^j$, and $J_i^j = H_i \cap K_e^j$. If t_i is odd, then $H_i = J_i$; we therefore assume t_i even and write t for t_i''. Lemma 6.2 lets us write $\chi|H_i^t$ as $\chi_0\chi_1$, where χ_0 extends the character $\chi_{0,i}$ of Theorem 6.1(8); χ_0 extends to a character of J_i^t, by Lemma 4.1, since G_i commutes with χ_0 and $J_i^t \subseteq (K_e \cap G_i)(H_i \cap K_e^{t_i'})$. We shall assume that $s_r = 0$; the calculations with $s_r > 0$ are similar, but easier.

Write $m = m_e^{f_{r-1}}(e_{r-1})$, $m^\times =$ set of elements of m whose image in A_e/A_e^1 is invertible; we use coset representatives for m, $m^\times$ as described in §1, but we will refine this later. Write $\eta = \eta_r = \eta_{r-1}$. Coset representatives for J_i^t/H_i^t are given by elements $1 + \alpha\eta^t$, where α runs through a $m_e^{f_i}(e_i)$-subspace V of $m_e^{f_{i-1}}(e_{i-1})$, and coset representatives for J_i/H_i by elements $\delta\eta^h(1 + \alpha\eta^t)$, where $\delta \in m^\times$, $h \in \mathbb{Z}$, and $\alpha \in V$. H_1^1 is normal in J_i.

We need two other facts before we start on the calculation:

(1) $\chi_0^\delta = \chi_0$ for $\delta \in m^\times$ (because $m^\times \subseteq G_i$), while $\chi_0^{\eta^{-h}}\chi_0^{-1} = \psi^{(h)}$ is a character on $H_i^t/H_i^{t_i}$. The $\psi^{(h)}$ are fixed by $m^\times$ because η^{-h} normalizes $m^\times$ (mod elements of H_i^1, which fix $\psi^{(h)}$).

(2) The alternating bilinear form B on V defined by $B(\alpha_1, \alpha_2) = \chi_1((1 + \alpha_1\eta^t), (1 + \alpha_2\eta^t)) = \chi_1(1 + (\alpha_1^{\sigma'}\alpha_2 - \alpha_2^{\sigma'}\alpha_1)\eta^{2t})$ is nondegenerate. (This is a calculation using Lemma 4.7.)

For $x \in J_i^1$, ρ^x and ρ are equivalent. In fact, a stronger statement holds: if $x = 1 + \alpha\eta^t$ is one of the cosets representatives in J_i^t, then Lemma 4.2 implies that $\rho^x(y) = \rho(y)$ for $y \in H_i^1$.

For the coset representatives $\delta\eta^h$, ρ is already defined. We extend ρ to a projective representation ρ_i on J_i by

$$\rho_i(x(1 + \alpha\eta^t)) = \chi_0(1 + \alpha\eta^t)\rho(x), \qquad x \in H_i, \ \alpha \in V.$$

If we knew that $\rho\big|_{H_i^1}$ were irreducible, this would be a projective representation by general nonsense. In our case, this needs to be checked, but it falls out of the following computation. (It can also be easily verified directly.) For

$y_1, y_2 \in H_i^1$, we compute:

$$\begin{aligned}&\rho_i(y_1\delta_1\eta^g(1+\alpha_1\eta^t))\rho_i(y_2\delta_2\eta^h(1+\alpha_2\eta^t))\\ &\qquad\qquad\qquad\cdot[\rho_i(y_1\delta_1\eta^g(1+\alpha_1\eta^t)y_2\delta_2\eta^h(1+\alpha_2\eta^t))]^{-1}\\ &\quad=\chi_0(1+\alpha_1\eta^t)\chi_0(1+\alpha_2\eta^t)\rho(y_1\delta_1\eta^g y_2\delta_2\eta^h)\\ &\quad\cdot[\rho_i(y_1\delta_1\eta^g y_2\delta_2\eta^h)\rho_i([y_2\delta_2\eta^h]^{-1}(1+\alpha_1\eta^t)y_2\delta_2\eta^h(1+\alpha_2\eta^t))]^{-1}.\end{aligned}\tag{A.1}$$

Write $y_2^{-1}(1+\alpha_1\eta^t)y_2 = z_0(1+\alpha_1\eta^t)$, with $z_0 \in H_i^{t+1}$; $\chi(z_0)=1$ from Lemma 4.2. Then $\chi([\delta_2\eta^h]^{-1}z_0\delta_2\eta^h)=1$, since $\delta_2\eta^h$ commutes with χ on H_i^{t+1}, and we can omit this term from the calculations. Hence

$$\rho_i([y_2\delta_2\eta^h]^{-1}(1+\alpha_1\eta^t)y_2\delta_2\eta^h(1+\alpha_2\eta^t)) = \rho_i([\delta_2\eta^h]^{-1}(1+\alpha_1\eta^t)\delta_2\eta^h(1+\alpha_2\eta^t)).$$

Write $[\delta_2 h^h]^{-1}(1+\alpha_1\eta^t)\delta_2\eta^h = u(\delta_2, h; \alpha_1)(1+\alpha_3\eta^t)$ with $\alpha_3 \in V$ and $u(\delta_2, h; \alpha_1) \in H_i^{t+1}$. Then the last expression is

$$\rho_i(u(\delta_2, h; \alpha_1))\rho_i(1+\alpha_3\eta^t)(1+\alpha_2\eta^t),$$

which is scalar. Thus (A.1) is scalar (and ρ_i is projective). Write

$$(1+\alpha_3\eta^t)(1+\alpha_2\eta^t) = (1+(\alpha_2+\alpha_3)\eta^t)z(\alpha_2, \alpha_3), \qquad z(\alpha_2, \alpha_3) \in H_i^t.$$

Then

$$\begin{aligned}\rho_i((1+\alpha_3\eta^t)(1+\alpha_2\eta^t)) &= \chi_0[(1+(\alpha_2+\alpha_3)\eta^t)z(\alpha_2, \alpha_3)]\chi_1[z(\alpha_2, \alpha_3)]I\\ &= \rho_i(1+\alpha_3\eta^t)\rho_i(1+\alpha_2\eta^t)\chi_1[z(\alpha_2, \alpha_3)],\end{aligned}$$

and

$$\rho_i(1+\alpha_3\eta^t) = \chi_0(1+\alpha_3\eta^t) = \chi_0(u(\delta_2, h, \alpha_1))^{-1}\psi^{(h)}(1+\alpha_1\eta^t).$$

So the scalar in (A.1) is the scalar in

$$[\rho_i(u(\delta_2, h, \alpha_1))\chi_0(u(\delta_2, h, \alpha_1))^{-1}\psi^{(h)}(1+\alpha_1\eta^t)\chi_1[z(\alpha_2, \alpha_3)]]^{-1};$$

that is,

$$[\chi_1(z(\alpha_2, \alpha_3))\psi^{(h)}(1+\alpha_1\eta^t)\chi_1(u(\delta_2, h; \alpha_1))]^{-1}.$$

Write $\xi(\delta_1, g, \alpha_1; \delta_2, h, \alpha_2) = \psi^{(h)}(1+\alpha_1\eta^s)\chi_1(u(\delta_2, h; \alpha_1)) = \chi^*(\delta_2\eta^h, \alpha_1)$. This is a 2-cocycle (here, (δ, g, a) represents $\delta\eta^g(1+\alpha\eta^t)$), since, as is easily checked, the other factor is also a 2-cocycle. We now analyze ξ.

First of all, we get rid of δ_1 and δ_2. We can write $\delta\eta^h = \eta^h\varepsilon$ for some $\varepsilon \in m^\times$. We now refine the choice of coset representatives. Let $\overline{m} \cong M_{f_i/f_{i-1}}(k_{f_{i-1}})$ be the image of m in A_e/A_e^1, and similarly for $\overline{m}^*$; let $\overline{\delta} \in \overline{m}$ correspond to $\delta \in m$; similarly, let $\overline{V} \cong$ a complement to $\overline{m}_e^{f_{i+1}}(e_{i+1})$ in $\overline{m}_e^{f_i}(e_i)$ (where the bars signify that we are working in the quotient spaces, not with coset representatives). We can choose $\overline{V}$ to be stable under the action of $\overline{m}^\times$ given by $\overline{\alpha} \mapsto \overline{\delta\alpha}(\overline{\delta}^{\sigma^t})^{-1}$, $\overline{\delta} \in \overline{m}^\times$. The reason is that we can regard $\overline{m}_e^{e_i}(f_i)$ as e/e_i copies of $M_{f/f_i}(k_{f_i})$ and $\overline{m}_e^{f_{i+1}}(e_{i+1})$ as e/e_{i+1} copies of $\overline{m}_{f/f_{i+1}}$, embedded in each (e_i/e_{i+1})th copy of $M_{f/f_i}(k_{f_i})$; we take the complement in a summand to be the complement under Tr when a copy is embedded and to be all of the summand when no copy is embedded, and sum. We now divide $\overline{V}$ into orbits under the above action and choose coset representatives in each orbit so that for some α in the orbit, the coset representatives for the other representatives are all of the form $\delta\alpha(\delta^{\sigma'})^{-1}$ for some $\delta \in m^\times$. It is not then true that for arbitrary $1+\alpha\eta^t$, we have $\delta(1+\alpha\eta^t)\delta^{-1} = 1 + \zeta\eta^t$ with $\zeta =$ coset representative for $\delta\alpha(\delta^{\sigma'})^{-1}$, but it is true that if ζ is that coset representative, then $\delta(1+\alpha\eta^t)(\delta^{\sigma'})^{-1} = \lambda(1+\zeta\eta^t)(\lambda^{\sigma'})^{-1}$ with $\lambda \in G_{r-1} \cap K_e^1$. (If $\beta = \delta_0\alpha\delta_0^{-1}$ with $\delta_0 \in m^\times$ and $\zeta = \delta_1\alpha\delta_1^{-1}$ with $\delta_1 \in m^\times$, take $\lambda = \delta_1\delta_0^{-1}$.) Now let $\eta^{-h}(1+\alpha\eta^t)\eta^h = (1+\alpha_0\eta^t)y$, where $\alpha_0 \in V$ and $y \in G_{i-1} \cap K^{t_i''}$, and suppose that $\varepsilon\alpha_0(\varepsilon^{\sigma'})^{-1} = \lambda\alpha_1(\lambda^{\sigma'})^{-1}$, where $\delta\eta^h = \eta^h\varepsilon$, $\alpha_1 \in V$, and $\lambda \in G_{r-1} \cap K_e^1$. Then

$$(\delta\eta^h)^{-1}(1+\alpha_0\eta^t)y\delta\eta^h = \lambda(1+\alpha_1\eta^t)\lambda^{-1} \cdot \varepsilon^{-1}y\varepsilon.$$

Since $\psi^{(h)}$ is stable under $m^\times$ and χ_1 is stable under $G_{r-1} \cap K_e^1$ (use Lemma 4.8 to show that χ and χ_0 are stable under $G_{r-1} \cap K_e^1$), it follows that

$$\xi(\delta_1, g, \alpha_1; \delta_2, h, \alpha_2) = \xi(g, \alpha_1; h, \alpha_2).$$

We shall write this last simply as $C(g, \alpha_1; h, \alpha_2)$. It is independent of g and α_2; sometimes we write $C(g, \alpha_1; h, \alpha_2) = \chi^*(h, \alpha_1)$.

The cocycle condition gives

$$\begin{aligned}\chi^*(h, \alpha_1+\alpha_2) &= C(g, \alpha_1+\alpha_2; h, \alpha_3)\\ &= C(g, \alpha_1; 0, \alpha_2)C((g, \alpha_1)\cdot(0, \alpha_2); h, \alpha_3)\\ &= C(g, \alpha_1; (0, \alpha_2)\cdot(h, \alpha_3))C(0, \alpha_2; h, \alpha_3)\\ &= \chi^*(h, \alpha_1)\chi^*(h, \alpha_2),\end{aligned}$$

and similarly

$$\chi^*(g, \alpha)\chi^*(h, \alpha^g) = \chi^*(g+h, \alpha),$$

where α^g is the element in V such that $1 + \eta^{-g}\alpha\eta^s\eta^g$ is represented by $1 + \alpha^g\eta^s$. So if we write

$$\chi^*(\eta^g, \alpha) = \varphi(g)(\alpha), \qquad \varphi : M/M_1 \to \overline{V}\widehat{\ },$$

then φ is a 1-cocycle. If φ is a 1-coboundary, then so is C, since if $\varphi(g) = \mu^g/\mu$, then $C(g, \alpha_1; h, \alpha_2) = \nu((g, \alpha_1)\cdot(h, \alpha_2))/\nu(g, \alpha_1)\nu(h, \alpha_2)$ if $\nu(g, \alpha) = \mu(\alpha)$. Thus we examine φ.

Since $\eta^{e_{r-1}}$ is a central element in $\mathrm{GL}_n(F)$ times an element of H_i^1, it is easy to check that φ has order dividing $e_{r-1} = e_r$. Because $\overline{V}\widehat{\ }$ is a p-group, standard theory shows that φ has order dividing p. This, of course, makes φ (hence C) trivial if $(p, e_r) = 1$. If $p | e_r$, we use the inverse of the other part of our cocycle,

$$C_0(\delta_1\eta^g(1+\alpha_1\eta^t), \delta_2\eta^h(1+\alpha_2\eta^t)) = \chi_1(z(\alpha_2, \alpha_3)).$$

This is the cocycle corresponding to the Weil representation, and

$$C_0(1+\alpha_1\eta^t, 1+\alpha_2\eta^t) = B(\alpha_1, \alpha_2)$$

is a nondegenerate bilinear form on $\overline{V} \times \overline{V}$. Hence there exists α_0 with

$$C_0(1+\alpha_0\eta^t, 1+\alpha\eta^t) = \varphi(1)(\alpha)^{-1} \quad \forall \alpha \in \overline{V}.$$

If we replace η by $\eta_* = \eta(1+\alpha_0\eta^t)$, then

$$C_0C(\delta_1\eta^g(1+\alpha_1\eta^t), \delta_2\eta^h(1+\alpha_2\eta^t)) = C_0(\delta_1\eta_*^g(1+\alpha_1\eta^t), \delta_2\eta_*^h(1+\alpha_2\eta^t)).$$

That is, the cocycle $(C_0C)^{-1}$ is the inverse of one for the Weil representation, but we need to use η_* instead of η to generate the cyclic element. Another way of saying this is that we are applying an outer automorphism to J_i/J_i^1. (If $2\nmid e$, we can use η_* as a substitute for η_i and redefine E_i, G_i accordingly. When $2|e$, this may not be possible because conjugation by η_* need not normalize F_{f_i}.) The outer automorphism depends only on χ, and not on all of σ. We now tensor with the Weil representation W to get an ordinary irreducible representation on J_i. Note that W is trivial on J_i^1, so that the induction hypotheses apply to $\sigma_1 \otimes W$. Since W is irreducible on K_i^t/H_i^t, any operator commuting with $\sigma_1 \otimes W$ must be of the form $A \otimes I$; as σ_1 is irreducible on H_i, A must be a multiple of I. Therefore $\sigma_1 \otimes W$ is irreducible. This concludes the construction.

References

1. C. Asmuth and D. Keys, *Supercuspidal representations of* GSp_4 *over a local field of residual characteristic* 2. I, II, preprints.
2. C. Bushnell, *Hereditary orders, Gauss sums, and supercuspidal representations of* GL_N, J. Reine Angew. Math. **375/376** (1987), 184–210.
3. H. Carayol, *Représentations cuspidales du groupe linéaire*, Ann. Sci École Norm. Sup. (4) **17** (1984), 191–225.
4. L. Corwin, *The unitary dual for the multiplicative group of arbitrary division algebras over local fields*, J. Amer. Math. Soc. **2** (1989), 565–598.
5. ——, *Constructing the supercuspidal representations of* $\mathrm{GL}_n(F)$, F *p-adic*, Proc. Bowdoin Conf. August 1989 (to appear).
6. L. Corwin, A. Moy, and P. J. Sally, *Degrees and formal degrees for division algebras and* GL_n *over a p-adic field*, Pacific J. Math. **141** (1990), 21–45.
7. P. Deligne, D. Kazhdan, and M.-F. Vigneras, *Représentations des algèbres centrales simples p-adiques*, Représentations des Groupes Rédutifs sur un Corps Local, Hermann, Paris, 1984, pp. 33–117.
8. P. Gérradin, *Construction de séries discrètes p-adiques*, Lecture Notes in Math., vol. 462, Springer, 1975.
9. R. Godement and H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Math., vol. 260, Springer, 1972.
10. R. Howe, *Tamely ramified supercuspidal representations of* GL_n, Pacific J. Math. **73** (1977), 365–381.
11. R. Howe and A. Moy, *Minimal K-types for* GL_n, preprint, 1986.
12. ——, *Hecke algebra isomorphisms for* GL_n *over a p-adic field*, preprint.
13. D. Kazhdan, *Representations of groups over close local fields*, J. Analyse Math. **47** (1986), 175–179.
14. H. Koch, *Eisensteinsche Polynomfolgen und Arithmetik in Divisionsalgebren über lokalen Körpern*, Math. Nachr. **104** (1981), 229–251.
15. P. Kutzko and D. Manderscheid, *On intertwining operators for* $\mathrm{GL}_N(F)$, F *a non-Archimedean p-adic field*, Duke Math. J. **57** (1988), 275–293.
16. ——, *On the supercuspidal representations of* GL_N, N *the product of two primes*, preprint, 1988.
17. G. W. Mackey, *Induced representations of groups*, Amer. J. Math. **73** (1951).
18. L. Morris, *P-cuspidal representations of level* 1, preprint.
19. A. Moy, *Local constants and the tame Langlands correspondence*, Amer. J. Math. **108** (1986), 863–930.
20. ——, *Representations of* $\mathrm{GS}_p(4)$ *over a p-adic field*. I, II, Compositio Math. **66** (1988), 237–284, 285–328.
21. J. Rogawski, *Representations of* $\mathrm{GL}(n)$ *and division algebras over a p-adic field*, Duke Math. J. **50** (1983), 161–196.

22. T. Springer, *Characters of special groups,* Semisimple Algebraic Groups and Related Finite Groups, Lecture Notes in Math., vol. 131, Springer, 1970, pp. 121–166.

23. A. V. Zelevinsky, *Induced representations of reductive p-adic groups,* II. *On irreducible representations of* GL(n) , Ann. Sci. École Norm. Sup. (4) **13** (1980), 165–211.

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